Civil Eng. 2 Mathematics: Second order PDEs: Separation of Variables.

This sheet can be found on the Web: http://www.ma.ic.ac.uk/~ajm8/Civ2

Classification of 2nd order PDEs in two variables

Most physical systems are governed by second order partial differential equations, or PDEs. Such equations fall into three basic types. Consider the equation for $u(x, y)$

$$a u_{xx} + b u_{xy} + c u_{yy} = f \; , \tag{1}$$

where the functions a, b, c and f do not depend on u_{xx}, u_{xy} or u_{yy}. They may, however, depend on x, y, u, u_x and u_y. The **Characteristic Equation** of (1) is

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a} \; . \tag{2}$$

Equation (1) is classified as hyperbolic, parabolic or elliptic according to:

$$\text{If} \quad \left\{ \begin{array}{lll} b^2 - 4ac > 0 & \text{2 real roots, (1) is} & \textbf{hyperbolic} \\ b^2 - 4ac = 0 & \text{1 real root, (1) is} & \textbf{parabolic} \\ b^2 - 4ac < 0 & \text{0 real roots, (1) is} & \textbf{elliptic}. \end{array} \right\} \; . \tag{3}$$

For hyperbolic equations, (2) is an ODE for $y(x)$ which can be integrated to define two sets of curves (one for the + sign, one for the −), called the **characteristics** of (1). Characteristics are **curves along which information travels at a finite speed.** They are associated with "time-like" behaviour, and a characteristic speed. In contrast elliptic problems have no "time-like" variable; x and y behave like space coordinates.

Hyperbolic Equations: A typical example is the **one-dimensional wave equation** for $u(x, t)$,

$$u_{tt} = c^2 u_{xx} \qquad \text{where } c \text{ is the constant wave speed.} \tag{4}$$

Hyperbolic equations should be solved with **two** initial conditions (at $t = 0$, say).

Elliptic Equations: These have no characteristics; no lines along which information travels. A typical elliptic equation is **Laplace's equation** for $u(x, y)$

$$\nabla^2 u \equiv u_{xx} + u_{yy} = 0 \qquad \text{in} \quad D, \tag{5}$$

where D is some region of (x, y)-space. This equation requires **one** boundary condition (say $u = f$) on the boundary of D.

Parabolic Equations: A typical example is the **diffusion equation** for $u(x, t)$

$$u_t = K u_{xx} \qquad \text{where } K > 0 \text{ is the constant diffusivity.} \tag{6}$$

Parabolic equations require **one** initial condition and it is vital that we move "forwards in time." Physically, parabolic equations describe the smoothing out of an initial configuration towards an equilibrium. Many different initial conditions give rise to almost the same final state. This is why running the process backwards in time is an ill-posed problem. You can't un-stir a cup of tea!

Summary

Equation type	Appropriate B.C.	Method of solution
Hyperbolic	2 Initial	Step in either direction from initial line
Parabolic	1 Initial	Step in one direction only from initial line
Elliptic	1 Boundary	Must solve everywhere simultaneously

Exact Solutions by "Separation of Variables"

Consider the example problem of the flow of heat in a bar,

$$\left.\begin{array}{rl} u_t = u_{xx} & \text{in} \quad 0 < x < 1,\ t > 0 \\ \text{with} \quad u(0,t) = u(1,t) = 0, & u(x,0) = u_0(x) \end{array}\right\} \tag{7}$$

We look for **separable solutions** of the PDE of the form $u(x,t) = X(x)T(t)$, so that

$$XT' = X''T \qquad \text{or} \quad \frac{T'}{T} = \frac{X''}{X} = -\omega^2, \qquad \text{say.} \tag{8}$$

As T'/T is a function of t only, while X''/X is a function of x only, both functions must be a constant, which we take to be negative. Then the functions $X(x)$ and $T(t)$ take the forms

$$X = A\cos\omega x + B\sin\omega x, \qquad \text{and} \quad T = Ce^{-\omega^2 t}\ . \tag{9}$$

If we require X to obey the boundary conditions in (7), namely $X(0) = X(1) = 0$, we obtain non-zero solutions only if $A = 0$ and $\omega = m\pi$, for some integer m, so that

$$u = B_m \sin(m\pi x)\, e^{-m^2\pi^2 t},$$

for some constant B_m. As (7) is a linear problem, we may combine solutions to obtain a more general solution in the form

$$u(x,\, t) = \sum_{m=1}^{\infty} B_m \sin(m\pi x)\, e^{-m^2\pi^2 t}\ . \tag{10}$$

The initial condition will be satisfied if

$$u(x,\, 0) = \sum_{m=1}^{\infty} B_m \sin(m\pi x) = u_0(x)\ . \tag{11}$$

Thus all we need do to obtain the solution of (7) is to expand the initial condition $u = u_0(x)$ in a **Fourier series**, and substitute the appropriate values of the constants B_m into (10). Using the orthogonality relations, we find

$$B_n = 2\int_0^1 u_0(x)\sin(n\pi x)dx\ .$$