We started here.

5. Reduce the partial differential equation

$$12\,\frac{\partial^2 u}{\partial x^2} - 7\,\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 u}{\partial y^2} + 4\,\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = e^{x+4y}$$

to its canonical form.

Solve the resulting equation and hence express the general solution of the original equation in terms of the variables

$$x + 3y \qquad \text{and} \qquad x + 4y\,.$$

Hence, or otherwise, show that the particular solution with

$$u\,(x,\,0) = x\,(1-x)\,e^{x}\;; \quad u\,(0,\,y) = 3y\,(1-4y)\,e^{4y}$$

is

$$u\,(x,\,y) = (x+3y)\,(1-x-4y)\,e^{x+4y}\,.$$

6. The function $f\,(x)$ is defined by

$$f\,(x) = x\,(1-x)\,.$$

Over the range $0 < x < 1$ the function can be represented by either a half-range Fourier cosine series:

$$\frac{1}{2}\,a_0 + \sum_{n=1}^{\infty} a_n\,\cos(n\pi x)$$

or a half-range Fourier sine series:

$$\sum_{n=1}^{\infty} b_n\,\sin(n\pi x)\,.$$

Show that $a_0 = \frac{1}{3}$ and determine the coefficients a_n and b_n for $n \geq 1$, distinguishing in your answer between odd and even values of n.

Use Parseval's theorem to deduce the value of

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^6}\,.$$

PLEASE TURN OVER

Let $\left.\begin{aligned} \xi &= x + ay \\ \eta &= x + by \end{aligned}\right\}$ for constants a, b.

$$\frac{\partial u}{\partial x} = \underset{=1}{\frac{\partial \xi}{\partial x}} \frac{\partial u}{\partial \xi} + \underset{=1}{\frac{\partial \eta}{\partial x}} \frac{\partial u}{\partial \eta}$$

$(x,y) \to (\xi, \eta)$.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}$$

$$\frac{\partial u}{\partial y} = \frac{\partial \xi}{\partial y} \frac{\partial u}{\partial \xi} + \frac{\partial \eta}{\partial y} \frac{\partial u}{\partial \eta}$$

$$= a \frac{\partial u}{\partial \xi} + b \frac{\partial u}{\partial \eta}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial x}\right)$$

$$= \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right) + \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right)$$

$$= \frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \cancel{\frac{\partial^2 u}{\partial \eta \partial \xi}} + \frac{\partial^2 u}{\partial \eta^2}$$

So $$\frac{\partial^2 u}{\partial x \partial y} = a\frac{\partial^2 u}{\partial \xi^2} + (a+b)\frac{\partial^2 u}{\partial \xi \partial \eta} + b\frac{\partial^2 u}{\partial \eta^2}$$

and $$\frac{\partial u}{\partial y^2} = a^2\frac{\partial^2 u}{\partial \xi^2} + 2ab\frac{\partial^2 u}{\partial \xi \partial \eta} + b^2\frac{\partial^2 u}{\partial \eta^2}$$

Sub:

$$12\left(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}\right)$$
$$-7\left(a u_{\xi\xi} + (a+b)u_{\xi\eta} + b u_{\eta\eta}\right)$$
$$+\left(a^2 u_{\xi\xi} + 2ab\, u_{\xi\eta} + b^2 u_{\eta\eta}\right)$$
$$+4\left(u_\xi + u_\eta\right) - \left(a u_\xi + b u_\eta\right) = e^{x+4y}$$

$$u_{\xi\xi}\left[12 - 7a + a^2\right] + u_{\xi\eta}\left[24 - 7(a+b) + 2ab\right]$$
$$+ u_{\eta\eta}\left[12 - 7b + b^2\right] + u_\xi(4-a) + u_\eta(4-b) = e^{x+4y}$$

Choose a and b s.t these vanish.

i.e $a = 3, b = 4$.

$$u_{\xi\eta}(24-7+7+2\times 12)$$

$$+u_\xi(4-3)+u_\eta(4-4)=e^{x+4y}$$

Or $u_{\xi\eta}(-1)+u_\xi(+1)=e^\eta$

$\left[\eta = x+by,\ b=4\right]$

$$\boxed{u_{\xi\eta} - u_\xi = -e^\eta}$$

Is the canonical form.

Write $w=u_\xi$

$$w_\eta + w = -e^\eta.$$

Integrating factor $e^{\int 1\,d\eta} = e^{-\eta}$

$$e^{-\eta}w_\eta + e^{-\eta}w = -1$$

$$\frac{d}{d\eta}\left[e^{-\eta}w\right] = -1$$

$$\Rightarrow w e^{-\eta} = -\cancel{\ldots}\,\eta + A(\xi)$$

↑ arbitrary function.

$$u_\xi = w = -\cancel{\ldots}\ \eta e^{\eta} + A(\xi)e^{+\eta}$$

Integrate w.r.t. ξ.

$$u = -\cancel{\ldots}\,\xi e^{\eta} + B(\xi)e^{+\eta} + C(\eta)$$

Look at given answer.

$$u = \underbrace{(x+3y)}_{\xi}(1-\eta)e^{\eta}.$$

Doesn't look quite right.

Now it is!

$$u = -\eta\xi e^{\eta} + B(\xi)e^{\eta} + C(\eta)$$

$\xi = x+3y \quad , \quad \eta = x+4y$

When $y=0$ $\underline{u = x(1-x)e^{x}}$

When $y=0$ $\eta = x = \xi$

$$u = -\cancel{x^2}e^{x} + B(x)e^{x} + C(x)$$

$$\parallel$$

$$xe^{x} - \cancel{x^2}e^{x}$$

Or $\underline{B(x)e^{x} + C(x) = xe^{x}}$

Also, when $x=0$ $\underline{u = 3y(1-4y)e^{4y}}$

When $x=0$ $\xi = 3y$ $\eta = 4y$

In fact

$$u = -\cancel{4y(3y)}e^{4y} + B(3y)e^{4y} + C(4y)$$

$$\parallel$$

$$3ye^{4y} - \cancel{12y^2}e^{4y}$$

$$\underline{B(3y)e^{4y} + C(4y) = 3ye^{4y}}$$

So $B(x) = x$ $C(x) = 0$. END.

$$u = f(x)\,g(y)$$

$$u_{xx} + u_{yy} = 0$$

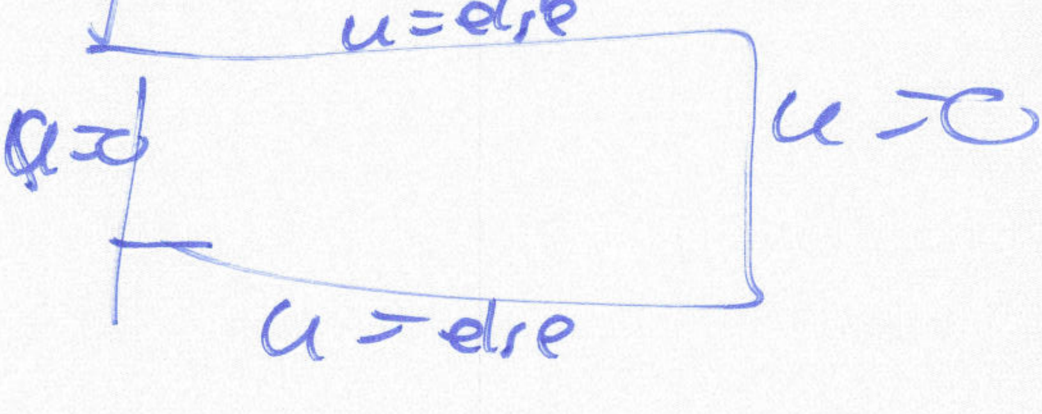
$$f''g + fg'' = 0$$

$$\frac{f''}{f}(x) = -\frac{g''}{g}(y)$$

$= \text{constant} < 0$

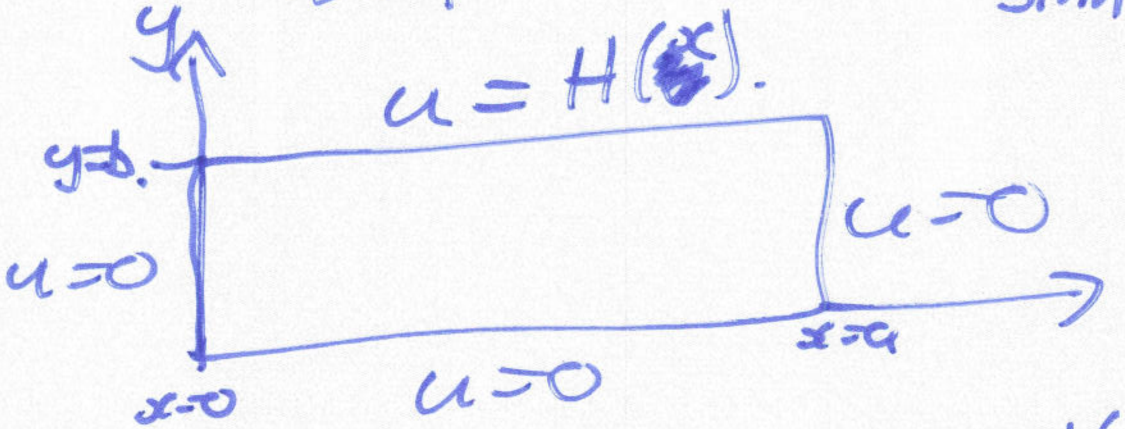
$f \sim$ sines

$g \sim$ exponentials.



$$u = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) \cancel{g_n(y)} \; ?$$

$$\sinh\left(\frac{n\pi y}{a}\right)$$



When $y=b$, we want $u = H(x)$

$\sin \frac{3\pi x}{a}$

$$H(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) \underline{\sinh\left(\frac{n\pi b}{a}\right)}.$$

How do we find A_n?

Multiply by $\sin\left(\frac{p\pi x}{a}\right)$ and integrate

where $p = 1, 2, 3 \ldots$

$$\int_0^a H(x) \sin\left(\frac{p\pi x}{a}\right) dx =$$

$$\int_0^a \sin\left(\frac{p\pi x}{a}\right)\left(\sum_{n=1}^{\infty} \ldots \ldots\right) dx.$$

$$= 0 + 0 + 0 + 0 + \int_0^a \sin^2\left(\frac{p\pi x}{a}\right) A_p \sinh(\;)\, dx$$

$$I = \int_{R} (x^4 - y^4)\, dx\, dy$$

Q8(2) Exam 2004
See later.

$u = xy \quad v = x^2 - y^2.$

$$= \int \Big(\qquad \Big) |J|^{-1} du\, dv$$

$$J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} y & x \\ 2x & -2y \end{vmatrix} = -2y^2 - 2x^2$$

$$I = \int_{-1}^{1} \int_{1}^{2} (x^4 - y^4) \frac{1}{2(x^2+y^2)} \quad du \quad dv$$

$$= \frac{1}{2} \int_{-1}^{1} \int_{1}^{2} (x^2 - y^2)\, du\, dv$$

$$= \frac{1}{2} \int_{-1}^{1} \int_{1}^{2} v \, du\, dv \quad = 0.$$

Imperial College
London

[II(CE) 2009]

M.ENG. EXAMINATIONS 2009

PART II : MATHEMATICS (CIVIL ENGINEERING)

Date Wednesday 3rd June 2009 2.00 - 5.00 pm

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.

Answer EIGHT questions: 6 from Section A (Questions 1-10) and 2 from Section B (Questions 11 - 13).

Answers to questions from Sections A and B should be written in separate answerbooks.

A mathematical formulae sheet is provided.

Statistical data sheets are provided.

[*Before starting, please make sure that the paper is complete; there should be EIGHT pages, with a total of THIRTEEN questions. Ask the invigilator for a replacement if your copy is faulty.*]

1. The matrix A is given by

$$A = \begin{pmatrix} 4 & 1 & 0 \\ 4 & 4 & 1 \\ 40 & 16 & 4 \end{pmatrix} .$$

By using Gaussian elimination, or otherwise, obtain the matrices L and U in the LU-factorisation: $A = LU$, where L is a lower triangular matrix with 1's along the main diagonal and U is an upper triangular matrix.

Hence obtain the inverse matrix A^{-1} and check that $AA^{-1} = I$, where I is the identity matrix.

2. Show that for a suitable value of n, the homogeneous ODE

$$x^2y'' - xy' + y = 0$$

has the solution $y = x^n$.

Hence find the solution to the inhomogeneous problem

$$x^2y'' - xy' + y = x \ln x$$

with $y(1) = 0$ and $y(2) = 0$.

PLEASE TURN OVER

3. The Trapezium Method for solving the ODE $y' = f(x, y)$ on the regular grid $x_n = nh$ for some steplength h and $n = 0, 1, 2, \ldots$ is

$$y_{n+1} = y_n + \tfrac{1}{2} h \left[f(x_n, y_n) + f(x_{n+1}, y_{n+1}) \right] .$$

Define the local truncation error for this scheme, E_n, and show that

$$E_n = -\tfrac{1}{12} h^3 y'''(x_n) + O(h^4) .$$

Use this method with a steplength $h = 0.25$ to find an approximation to $y(1)$ for the problem

$$y' = 1 - y, \qquad y(0) = 2 .$$

Find the exact solution to this problem and show that

$$y(1) - y_4 \simeq 0.00193 \qquad \text{to three significant figures.}$$

4. A clarinet can be considered as a vibrating thin pipe of length L, open at one end and closed at the other. Air flow in a clarinet is governed by the equation for $u(x, t)$

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} ,$$

where c is the constant sound speed, t is time and x denotes distance from the closed end.

The boundary conditions on u are

$$u(0, t) = 0 \qquad \text{and} \qquad \frac{\partial u}{\partial x}(L, t) = 0 .$$

Use the method of separation of variables to find the general solution for u.

The *vibration frequencies* are the positive values of ω such that a possible solution is $u = f(x)\cos(\omega t)$. Show that the second smallest vibration frequency is 3 times the smallest.

Find the particular solution satisfying the initial conditions

$$u(x, 0) = 0 \qquad \text{and} \qquad \frac{\partial u}{\partial t}(x, 0) = \sin\left(\frac{3x}{2L}\right) .$$

5. The function $f(x)$ has period 2π, and over the range $-\pi \leq x \leq \pi$ it is defined by

$$f(x) = x - \frac{1}{\pi^2} x^3 .$$

The Fourier Series expansion of $f(x)$ is of the form

$$f(x) = \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)] .$$

Calculate the coefficients a_n and b_n of the Fourier series.

By using Parseval's theorem deduce the value of the infinite sum

$$S = \sum_{n=1}^{\infty} \frac{1}{n^6} .$$

6. Let $f(z) = u + iv$ be an analytic function of the complex variable $z = x + iy$. Its real and imaginary parts, u and v, satisfy linear relations called Cauchy-Riemann equations. Derive these equations by exploiting the fact that the value of the derivative

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

does not depend on the way Δz approaches zero .

Hint: *Use* $\Delta z = \Delta x$ *and* $\Delta z = i\Delta y$.

Deduce that both u and v satisfy the Laplace equation.

Find the (positive) value of the parameter λ such that

$$u(x, y) = e^{x^2 - y^2} \cos(\lambda xy)$$

could be the real part of an analytic function $f(z)$.

Using the Euler formula obtain the respective imaginary part v and reconstruct the function $f(z)$.

PLEASE TURN OVER

7. The temperature $T(x,\, t)$ of a metal rod of length $2L$ satisfies the heat conduction equation

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}\,, \quad -L \le x \le L\,, \quad t \ge 0\,,$$

with insulating boundary conditions $\dfrac{\partial T}{\partial x} = 0$ on $x = \pm L$.

The rod has an initial temperature distribution

$$T(x,\, 0) = \begin{cases} 0\,, & -L \le x < -L/2\,, \\ T_0, & -L/2 \le x \le L/2\,, \\ 0\,, & L/2 < x \le L\,, \end{cases}$$

where T_0 is a constant.

Use the method of separation of variables to obtain the temperature distribution $T(x,\, t)$ at all times in the form of a Fourier series expansion.

Hint: the initial temperature distribution is even and it remains so at all times.

Hence deduce the limiting value $\lim\limits_{t\to\infty} T\,(x,\, t)$.

8. (i) The region S is the sector of a disc lying between the lines $y = x$, $y = -x$ and the circle $x^2 + y^2 = a^2$ with $y \geq 0$.
Find the coordinates $(\overline{x}, \overline{y})$ of the centroid of S and the value I_{oz} of the moment of inertia about the z-axis of S.

Note: The coordinates of the centroid are given by

$$\overline{x} = \frac{1}{A} \iint_S x \, dx \, dy \, ,$$

$$\overline{y} = \frac{1}{A} \iint_S y \, dx \, dy \, ,$$

where A is the area of S.

The moment of inertia is given by

$$I_{0z} = \iint_S (x^2 + y^2) \, dx \, dy \, .$$

(ii) The integral $\iint_R (x^4 - y^4) \, dx \, dy$ is to be evaluated over the region R bounded by the four hyperbolae $xy = 1$, $xy = 2$, $x^2 - y^2 = 1$ and $x^2 - y^2 = -1$ with $x > 0$ and $y > 0$.

Sketch the region R and find the values of the new variables $u = xy$ and $v = x^2 - y^2$ on each part of the boundary of R.

Evaluate the integral by first changing to the new variables, and using the Jacobian for the transformation.

PLEASE TURN OVER

9. Show for any twice differentiable scalar field $\phi(x, y, z)$ and any twice differentiable vector field $\boldsymbol{E}(x, y, z)$ that

$$\text{curl grad}\,\phi = 0 \quad \text{and} \quad \text{div curl}\,\boldsymbol{E} = 0\,.$$

Suppose that the vector field $\boldsymbol{E}(x, y, z)$ is given by

$$\boldsymbol{E} = (x^2 - y^2)\boldsymbol{i} + (axy + z^2)\boldsymbol{j} + (byz + z)\boldsymbol{k}\,.$$

Find the values of the constants a and b that make $\text{curl}\,\boldsymbol{E} = 0$.

For these values of a and b find the potential $\phi(x, y, z)$ that satisfies

$$\boldsymbol{E} = \text{grad}\,\phi\,.$$

Note: $\text{grad}\,\phi \equiv \nabla\phi$, $\text{curl}\,\boldsymbol{E} \equiv \nabla \times \boldsymbol{E}$, $\text{div}\,\boldsymbol{E} \equiv \nabla \cdot \boldsymbol{E}$.

10. Write the ordinary differential equation

$$\ddot{x} + 2\dot{x} + 5x = 0$$

in the matrix form

$$\frac{d\boldsymbol{X}}{dt} = A\boldsymbol{X}\,,$$

where

$$\boldsymbol{X} = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}.$$

By calculating eigenvalues and eigenvectors of the matrix A find a solution $\boldsymbol{X}(t)$ that satisfies the initial condition

$$\boldsymbol{X}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Section B

11. The continuous random variable Y has density function $f(y) = ky(2+y)$ with $0 \leqslant y \leqslant 5$, where k is a constant.

   (i) Show that $k = \frac{3}{200}$ and calculate the mean and variance of Y.

   (ii) Calculate the probabilities of the events $A = \{0.5 \leqslant Y \leqslant 1\}$ and $B = \{0.8 \leqslant Y \leqslant 1.2\}$

   (iii) Calculate the probability $\mathrm{P}(A|B)$

   Calculate all answers to 3 decimal places.

12. The table below gives the discrete bivariate distribution of X_1 and X_2.

   (i) Write down the marginal probability distributions of X_1 and X_2, and calculate $\mathrm{E}(X_1)$ and $\mathrm{var}(X_1)$.

   (ii) What is the conditional distribution of X_2 given that $X_1 = 2$?

   (iii) What is the conditional distribution of X_2 given that $X_1 \neq 2$?

   (iv) Calculate $\mathrm{P}(X_1 > X_2)$, $\mathrm{P}(X_1 < X_2)$ and $\mathrm{P}(X_1 = X_2 \mid X_1 \leq X_2)$.

	$X_1 = 1$	2	3
$X_2 = 1$	0.10	0.06	0.14
2	0.10	0.04	0.06
3	0.20	0.10	0.20

13. (i) Slippage tests are carried out on five supplied specimens for a paving surface, resulting in the following observed values in arbitrary units: 4.4, 4.7, 5.2, 5.0, 4.7. Assuming that the observations form a random sample from a normal distribution, test the supplier's claim that the underlying mean value is 5.0.

   (ii) Two types of tiling are supplied for a walkway and some breaking-strength tests yield the following data in arbitrary units.

   Type A: 3.7 4.2 4.0 4.1 4.0
   Type B: 3.5 3.7 3.6 3.6 3.8 4.0

   Perform a test of the hypothesis that the underlying mean breaking-strengths are equal for Types A and B.

   State the assumptions required for validity of your test.

END OF PAPER