Civil Engineering 2: Mathematics

Autumn Term Progress Test
Wednesday 9th December 2009 11.00 - 12.00
Answer all questions

Question 1. Find the Fourier series of the 2π-periodic function $f(x)$ which is given by $f(x) = x$ in $-\pi < x \leq \pi$.

Discuss the sum of the series when $x = \pi$.

Question 2. The function $u(x,\, t)$ satisfies the PDE

$$u_t = u_{xx} \qquad \text{with} \quad u(0, t) = u(L, t) = 0.$$

Use the method of separation of variables to write down the general solution to this equation which decays to zero as $t \to \infty$.

Question 3. Determine whether or not the function $u(x, y) = x^2 y^6$ is the real or imaginary part of an analytic function of $z = x + iy$.

What equation is satisfied by the family of curves orthogonal to the family $u(x, y) = C$?

Question 4. The function $\phi(x,\, y) = x^3 - 3xy^2$.

$$\text{Calculate} \quad \text{(i) } \mathbf{u} \equiv \nabla\phi, \qquad \text{(ii) } \nabla \cdot \mathbf{u}, \qquad \text{(iii) } \nabla \times \mathbf{u}.$$

Write down a vector field which is perpendicular to the curves $\phi = \text{constant}$.

Question 5. A quarter of a circle, enclosed by the curves $x = y$, $x = -y$ and $x^2 + y^2 = R^2$, has a constant mass density ρ. Calculate the coordinates $(\bar{x}, \bar{y})$ of its centre of gravity.

Question 6. Calculate the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{bmatrix}.$$

Hence obtain the unitary matrix U such that $U^T A U = \text{diag}(\lambda_1, \lambda_2)$ and calculate A^4.

END OF TEST

Civ2 Winter Test model solutions

Question 1.

We note $a_n = 0$ for all n as the function $f(x)$ is odd.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)dx = \frac{2}{\pi n}\Big[-x\cos(nx)\Big]_0^{\pi} + \frac{2}{\pi n}\int_0^{\pi}\cos(nx)dx$$

$$= -\frac{2\pi}{\pi n}\cos(\pi n) + \frac{2}{\pi n^2}\sin(nx)|_0^{\pi} = \frac{2(-1)^{n+1}}{n} \ .$$

So the Fourier series is

$$f(x) = x = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx) \ .$$

At $x = \pi$, the function $f(x)$ is not continuous. Clearly, the series sums to 0 which is the average of the two values π and $-\pi$ either side of the jump. [**4 marks**].

Question 2.

Write $u(x, y) = A(t)B(x)$ then $A'B = AB''$ and

$$\frac{A'}{A} = \frac{B''}{B} = -\lambda^2 \ ,$$

where we take $\lambda^2 > 0$ to have a solution decaying in time, $A(t) \sim e^{-\lambda^2 t}$. For the function $B(x)$ we have the equation

$$B'' + \lambda^2 B = 0 \ ,$$

the general solution to which is

$$B(x) = \alpha\cos(\lambda x) + \beta\sin(\lambda x) \ .$$

The boundary conditions read $B(0) = \alpha = 0$ and $B(L) = \beta\sin(\lambda L)$, and so

$$\lambda = \lambda_n = \pi n/L \ , \quad n = 1, 2, \ldots \ ,$$

and the general solution is

$$u(x, t) = \sum_{n=1}^{\infty}\beta_n\sin(\pi n x/L)e^{-(\pi n/L)^2 t}.$$

[**4 marks**]

Question 3.

We have $u_x = 2xy^6$, $u_y = 6x^2y^5$, then $u_{xx} = 2y^6$ and $u_{yy} = 30x^2y^4$. So the Laplace equation

$$u_{xx} + u_{yy} = 2y^4(y^2 + 15x^2) \neq 0$$

is not satisfied in any finite area of the complex plane (just on the line $y = 0$) and therefore $u(x, y)$ can not be a real (or imaginary) part of an analytic function of $x + iy$.

The slope m_1 of the trajectories $u(x, y) = C$ is calculated from $du = u_x dx + u_y dy = 0$ and is given by

$$m_1 = \frac{dy}{dx}|_1 = -\frac{u_x}{u_y} = -\frac{y}{3x} .$$

The slope for the orthogonal family is therefore

$$m_2 = \frac{dy}{dx}|_2 = -\frac{1}{m_1} = \frac{3x}{y} .$$

Integrating $\int ydy - 3\int xdx$ we obtain that the orthogonal family of trajectories must satisfy the equation

$$y^2 - 3x^2 = C' .$$

[4 marks]

Question 4.

(i) We have $\mathbf{u} = \nabla\phi = (3x^2 - 3y^2)\mathbf{i} - 6xy\mathbf{j}$.

(ii) $\nabla \cdot \mathbf{u} = (3x^2 - 3y^2)_x - (6xy)_y = 0$.

(iii) $\nabla\times\mathbf{u} = \mathbf{k}\,[(-6xy)_x - (3x^2 - 3y^2)_y] = \mathbf{k}(-6y+6y) = 0$. Alternatively, they may quote that $\nabla \times \nabla\phi = 0$ for any ϕ.

(iv) the vector $u = \nabla\phi$ is normal to the curve $\phi =$ constant.

[4 marks]

Question 5.

The total mass of the object is $M = \rho A$ where A is the area. Use polar coordinates with appropriate Jacobian

$$x = r\cos\theta, \quad y = r\sin\theta, \quad J(r,\theta) = r$$

to compute

$$\iint_A dxdy = \int_0^R r dr \int_{\pi/4}^{3\pi/4} d\theta = \frac{\pi}{4}R^2 \ .$$

[Alternatively the area of a quarter circle is equal to a quarter of an area of a circle.]

Once the total mass is known, the centre of gravity is given by

$$M(\bar{x},\bar{y}) = \rho\left(\iint_A x\,dxdy, \iint_A y\,dxdy\right) \ .$$

Obviously $\bar{x} = 0$ as the object is symmetric in $x \to -x$ (or calculate the integral). For the other coordinate

$$M\bar{y} = \rho\int_0^R r^2 dr \int_{\pi/4}^{3\pi/4} \sin\theta\,d\theta = \rho\frac{1}{3}r^3|_0^R \cos\theta|_{3\pi/4}^{\pi/4} = \rho\frac{\sqrt{2}}{3}R^3$$

and so the coordinates of the centre of gravity are

$$(\bar{x},\bar{y}) = \left(0, \frac{4\sqrt{2}R}{3\pi}\right) \ .$$

[4 marks]

Question 6.

The eigenvalues follow from the determinant:

$$\begin{bmatrix} \frac{3}{2}-\lambda & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2}-\lambda \end{bmatrix} = \left(\lambda - \frac{3}{2}\right)^2 - \frac{1}{4} = 0, \quad \Rightarrow \quad \lambda = \frac{3}{2} \pm \frac{1}{2} = 1, 2 \ .$$

The first eigenvector ($\lambda = 1$) is

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0, \quad \Rightarrow \quad \mathbf{x} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} ,$$

and the second eigenvector ($\lambda = 2$) is

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 0, \quad \Rightarrow \quad \mathbf{y} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} .$$

The unitary matrix therefore is

$$U = (\mathbf{x}, \mathbf{y}) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} , \quad U^T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} .$$

It is readily verified that $U^T A U = \mathrm{diag}(1, 2)$. To compute A^4 write $A = U\mathrm{diag}(1, 2)U^T$ and so

$$A^4 = U\mathrm{diag}(1, 16)U^T = \frac{1}{2} \begin{bmatrix} 17 & 15 \\ 15 & 17 \end{bmatrix} .$$

[**4 marks**]