Civ. Eng. 2: Numerical solution of ODEs: Euler's method

This sheet can be found on the Web: http://www.ma.ic.ac.uk/∼ajm8/Civ2

Most differential equations are **non-linear**, and can rarely be solved exactly. However, it is usually possible to obtain arbitrarily accurate approximate solutions on computers. We will now discuss various methods for doing this.

As we are dealing with **ODEs**, there is only one independent variable, which we call x. All systems of ODEs can then be expressed in terms of a vector of unknowns, $\underline{\mathbf{y}}(x)$ which obeys the equation

$$\frac{d\underline{\mathbf{y}}}{dx} = \underline{\mathbf{f}}(x,\, \underline{\mathbf{y}}(x)) \qquad \text{where } \underline{\mathbf{f}} \text{ is a vector of known functions.}$$

We shall concentrate on the one-dimensional problem, as most of the methods we consider can easily be generalised to more dimensions. Suppose therefore that $y(x)$ solves the problem

$$\frac{dy}{dx} = f(x,\, y(x)) \qquad \text{with} \quad y(a) = b\ . \tag{1}$$

Finite Difference Methods

It is important to realise that most ODEs do not have a solution in terms of simple functions. However, they will have solutions which can be expressed as graphs, say. How can we obtain an approximation to the curves on these curves? A simple idea is to try to find some points which lie on, or close to, these graphs and join them up. So choose a **step-length** h, and define the values $x_n = a + nh$, where $n = 0,\, 1,\, 2\ldots$ The exact solution at the value $x = x_n$ can be written $y(x_n)$. We shall now seek some values y_n which we hope will approximate the real values. To do this we must translate the differential equation into an approximate equation valid at the discrete points $\{x_n\}$.

We recall the Taylor series for the function $y(x + h)$:

$$y(x+h) = y(x) + hy'(x) + \tfrac{1}{2}h^2y''(x) + \tfrac{1}{6}h^3y'''(x) + \tfrac{1}{24}h^4y''''(x) + \ldots$$

Assuming h is small, we can therefore approximate the derivative $y'(x)$ by

$$y'(x) = \frac{y(x+h) - y(x)}{h} - \tfrac{1}{2}hy''(x) + O(h^2)\ .$$

Here $O(h^2)$ denotes things which tend to zero as $h \to 0$ at least as fast as h^2. Now if we take $x = x_n$, then by definition $x + h = x_n + h = x_{n+1}$ and so

$$y'(x_n) = \frac{y(x_{n+1}) - y(x_n)}{h} + O(h) \simeq \frac{y_{n+1} - y_n}{h}$$

assuming that y_n is a good approximation to $y(x_n)$. We can use this last approximation in (1), to define the values y_n as the solution to

$$\frac{y_{n+1} - y_n}{h} = f(x_n,\, y_n) \qquad \text{or} \quad y_{n+1} = y_n + hf(x_n,\, y_n) \qquad \text{with} \quad y_0 = b\ . \tag{2}$$

This last relation is known as **Euler's (forward) method**.

An alternative idea is to regard $(y_{n+1} - y_n)/h$ as an approximation not for $y'(x_n)$ but instead for $y'(x_{n+1})$. This gives rise to the scheme known as **Euler's backward method:**

$$y_{n+1} = y_n + hf(x_{n+1}, y_{n+1}) \tag{3}$$

Note that if y_n is known and y_{n+1} is to be found this method is not quite ao simple to use, as it doesn't give an **explicit** formula for y_{n+1} in terms of known quantities.

Euler's method is very simple to use, and can be used to solve second order (or higher) ODEs also. As an illustration, consider the Simple Harmonic Motion problem

$$y'' = -y \qquad \text{with} \quad y(0) = 1, y'(0) = 0\ . \tag{4}$$

This problem has the exact solution $y = \cos x$. We start by writing this as two first order equations. Let $u = y$ and $v = y'$, so that the exact solution is $u = \cos x$, $v = -\sin x$. We note $u^2 + v^2 = 1$ for all x. Now (4) is equivalent to

$$u' = v, \qquad v' = -u \qquad \text{with} \quad u(0) = 1,\ v(0) = 0\ . \tag{5}$$

We can use Euler's forward method for these equations to give

$$u_{n+1} = u_n + hv_n, \qquad v_{n+1} = v_n - hu_n, \quad u_0 = 1,\ v_0 = 0. \tag{6}$$

It is a very simple matter to calculate u_n and v_n for a given h for as many n as we choose: see the computer demonstration. We note by simple algebra that

$$u_{n+1}^2 + v_{n+1}^2 = (1 + h^2)(u_n^2 + v_n^2)\ . \tag{7}$$

For comparison, Euler's **backward** method would give the equations

$$u_{n+1} = u_n + hv_{n+1}, \qquad v_{n+1} = v_n - hu_{n+1}, \quad u_0 = 1,\ v_0 = 0. \tag{8}$$

Solving these simultaneous equations we find

$$u_{n+1} = (u_n + hv_n)/(1 + h^2), \qquad v_{n+1} = (v_n - hu_n)/(1 + h^2)\ , \tag{9}$$

We can now step these equations forward in x. This time we see that

$$u_{n+1}^2 + v_{n+1}^2 = (u_n^2 + v_n^2)/(1 + h^2)\ , \tag{10}$$

We see that neither Euler's forward or backward methods conserves the **energy** of the system. If we want to solve for a long while we may get inaccurate results.