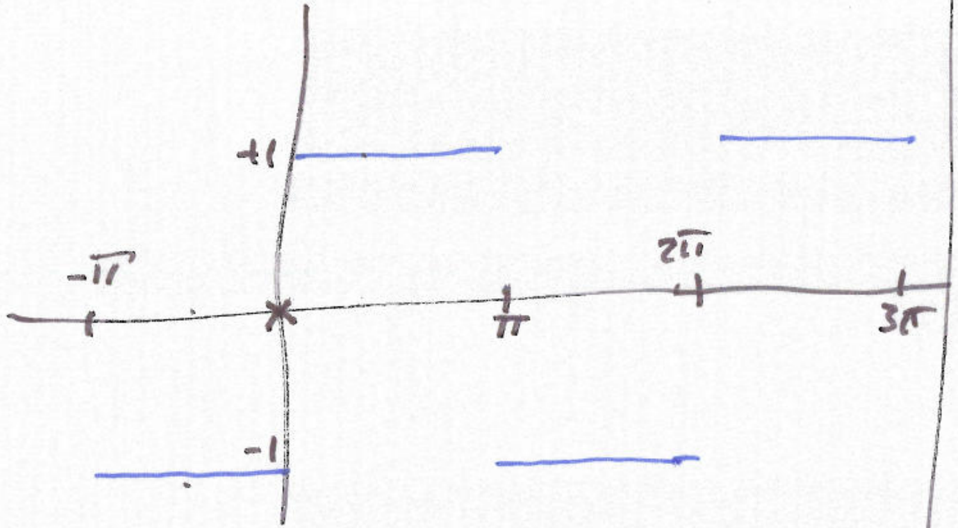
Fourier Series – do they really work?

Try some examples



Square wave:

$f(x) = +1 \qquad 0 < x < \pi$

$f(x) = -1 \qquad -\pi < x < 0$

$f(0)$, ~~f(−π)~~ $f(\pi)$ not yet ~~determined~~ defined.

Calculate a_n & b_n.

$L = \pi$, in above formula.

~~a_n~~

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$= \frac{1}{\pi}\left(\int_{-\pi}^{0} f(x)\cos nx\,dx + \int_{0}^{\pi} f(x)\cos nx\,dx\right)$$

$$= \frac{1}{\pi}\left(\int_{-\pi}^{0} (-1)\cos nx\,dx + \int_{0}^{\pi} (1)\cos nx\,dx\right)$$

$$= \frac{1}{\pi}\left[-\frac{1}{n}\sin nx\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_{0}^{\pi}$$

(assuming $n \neq 0$)

$$= 0$$

$$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} (-1)\,dx + \int_{0}^{\pi} (1)\,dx\right]$$

$$= \frac{1}{\pi}\left[-x\right]_{-\pi}^{0} + \frac{1}{\pi}\left[x\right]_{0}^{\pi} = \frac{1}{\pi}(-\pi + \pi)$$

$$= 0.$$

Similarly,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left(\int_{-\pi}^{0}(-1)\sin nx\,dx + \int_{0}^{\pi}(1)\sin nx\,dx\right)$$

$$= \frac{1}{\pi}\left(\left[\frac{\cos nx}{n}\right]_{-\pi}^{0} + \left[-\frac{\cos nx}{n}\right]_{0}^{\pi}\right)$$

$$= \frac{1}{\pi n}\left(1-\cos(-n\pi) - (\cos(n\pi)-1)\right)$$

$$= \frac{1}{\pi n}\left(2 - 2\cos(n\pi)\right)$$

$$= \frac{2}{\pi n}\left(1-\cos(n\pi)\right) = \frac{2}{\pi n}\left(1-(-1)^n\right)$$

Now $\cos(n\pi) = (-1)^n$

Or

$$b_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{4}{\pi n} & \text{if } n \text{ is odd} \end{cases}$$

So the F.S. for $f(x)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$\left.\begin{matrix} +1 \\ -1 \end{matrix}\right\} = \sum_{\substack{n=1 \\ (n\ \text{odd})}}^{\infty} \frac{4}{\pi n}\sin(nx)$$

$$\boxed{a_q = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos qx\,dx}$$

$q = 0, 1, 2, 3 \ldots\ldots$

because

$$\boxed{\int_0^{2\pi} \cos mx \cos nx\,dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \neq 0. \\ 2\pi & \text{if } m = n = 0 \end{cases}}$$

Also find.

$$\int_0^{2\pi} \sin(mx)\cos(nx)\,dx = 0.$$

m, n integers > 0.

The general case.

Suppose we have a $2L$-periodic function, defined between $x = -L$ and $x = +L$.

Then we can write:

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right]$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

The computer indicates that the Fourier Series converge to the function $f(x)$ [but with overshoots at any discontinuity].

Why was $a_n = 0$ for this function?

It is because the function $f(x)$ was odd i.e

$f(-x) = -f(x)$ for all x.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$\begin{aligned} f(-x) &= \sum b_n \sin(-nx) \\ &= -\sum b_n \sin nx \\ &= -f(x). \end{aligned}$$

Similarly if

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

Then $f(-x) = f(x)$

So $f(x)$ is even.

Even functions cosines only
Odd functions sines only