Transformation to a simpler form.

Just as with integrals, a suitable substitution may lead to a simpler problem, e.g.

$$x^2 y'' + x y' + \left(x^2 - \tfrac{1}{4}\right) y = x^{5/2}$$ ← Substitute in.

Seek a change of dependent variable (y) to simplify things:

Guess:

$$y = x^\alpha u(x).$$

where α is to be determined later for convenience.

Then
$$y = x^\alpha u$$
$$y' = \alpha x^{\alpha-1} u + x^\alpha u'$$
$$y'' = \alpha(\alpha-1) x^{\alpha-2} u + \alpha x^{\alpha-1} u' + \alpha(x^{\alpha-1}) u' + x^\alpha u''$$
$$= \alpha(\alpha-1) x^{\alpha-2} u + 2\alpha x^{\alpha-1} u' + x^\alpha u''$$

$$x^2\left[\alpha(\alpha-1)x^{\alpha-2}u + 2\alpha x^{\alpha-1}u' + x^\alpha u''\right] + x\left[\alpha x^{\alpha-1} u + x^\alpha u'\right] + \left(x^2 - \tfrac{1}{4}\right) x^\alpha u = x^{5/2}.$$

Or

$$x^{\alpha+2} u'' + u'\left[x^{\alpha+1}(2\alpha+1)\right] + u\left[x^\alpha\underbrace{\left[\alpha(\alpha-1) + \alpha - \tfrac{1}{4}\right]}_{\left[\alpha^2 - 1/4\right]} + x^{\alpha+2}\right] = x^{5/2}$$

We see that $\alpha^2 = 1/4$ simplifies the last term and $2\alpha + 1 = 0$ also if $\alpha = -1/2$.

So choose $\underline{\alpha = -1/2}$.

Get

$$x^{3/2} u'' + u'[0] + u[0 + x^{3/2}] = x^{5/2}$$

Or

$$x^{3/2}(u'' + u) = x^{5/2}$$

Or $\underline{u'' + u = x}$.

<u>C.F</u> $u'' + u = 0$.

We should <u>know</u> this. Else we cannot be considered remotely scientifically literate, let alone a renowned and widely respected Civil Engineer in 3 years time:

[Auxiliary equation $u = e^{mx}$, $m^2 + 1 = 0$, $m = \pm i$, $u = Ae^{ix} + Be^{-ix}$
$= C\cos x + D\sin x$

where $C = A + B$ $\quad D = i(A - B)$].

So $u = C\cos x + D\sin x$ is C.F.

<u>P.I.</u>: A solution is (by inspection) $u = x$

<u>G.S.</u>: $\underline{u = x + C\cos x + D\sin x}$

Now $y = x^{-1/2} u$

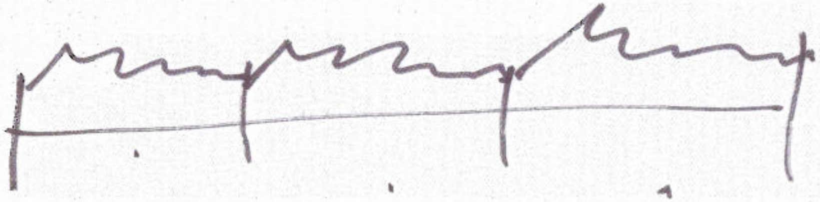
So

$$y = \underbrace{x^{1/2}}_{\text{P.I}} + \underbrace{\frac{C\cos x + D\sin x}{x^{1/2}}}_{\text{C.F.}}$$

is the solution.

If we have 2 pieces of data (e.g $y(1) = 0$
$y'(1) = 0$.

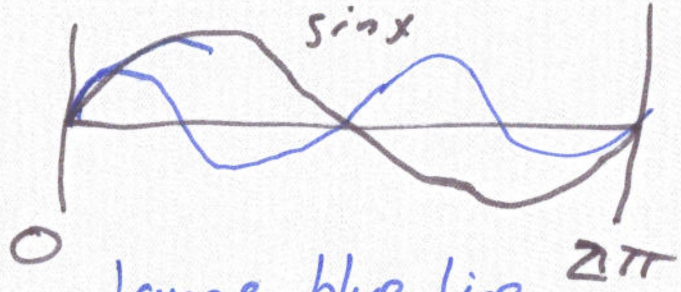
we can determine C & D and get the <u>unique</u> solution.



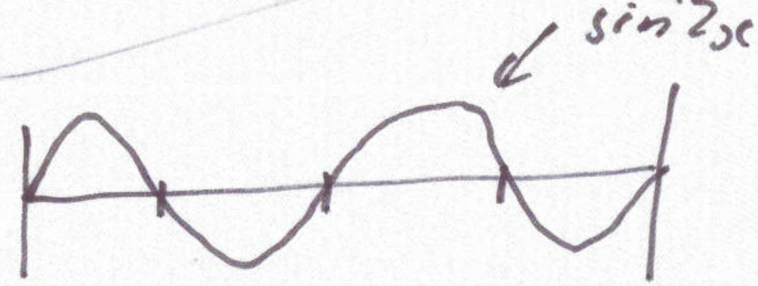
Fourier Series

Very imp~~or~~tant topic in Physics & Engineering.

<u>Idea:</u> Given a <u>periodic</u> function (e.g. heartbeat, tides) can it be represented in terms of the <u>natural</u> periodic functions $\sin x$, $\cos x$, $\sin 2x$, $\cos 2x$?



Ignore blue line.



etc.

Can we write a ~~[illegible]~~ 2π-periodic function

$$f(x) = \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \, ?$$

Fourier Series

A 2π-periodic function $f(x)$ is defined by

$$f(x+2\pi) = f(x) \text{ for all } x.$$

Can we write $f(x)$ as a "Fourier Sine series"

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

for suitable constants b_n?

(If so, b_n is called a "Fourier coefficient".)

How can we find $b_1, b_2, b_3 \ldots$?

Recall from A-level that

$$2 \sin mx \sin nx = \cos(mx - nx) - \cos(mx + nx)$$

(Derivation) ↓

$$\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}$$

Derivation #2 ↓

$$\cos(x+y) = \cos x \cos y - \sin x \sin y.$$

$$\cos(x-y) = \cos x \cos y + \sin x \sin y.$$

$$\cos(x+y) - \cos(x-y) = -2 \sin x \sin y.$$

Now put $x = \frac{A+B}{2}$, $y = \frac{A-B}{2}$ etc.

So what is

$$2\int_0^{2\pi} \sin(mx)\sin(nx)\,dx \;?$$

$$= \int_0^{2\pi} \left[\cos(m-n)x - \cos(m+n)x\right] dx$$

$$= \left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n} \right]_0^{2\pi}$$

$= 0$ provided m, n are integers.

and provided $m \neq n$
$(m \neq -n)$

If $m = n$.

$$2\int_0^{2\pi} \sin^2(nx)\,dx$$

$$= \int_0^{2\pi} (1 - \cos 2nx)\,dx$$

$$= \left[x - \frac{\sin(2nx)}{2n} \right]_0^{2\pi}$$

$$= 2\pi.$$

$$\boxed{\int_0^{2\pi} \sin(mx)\sin(nx)\,dx = \begin{cases} \pi & \text{if } m=n \\ 0 & \text{if } m \neq n \end{cases}}$$

$(m, n > 0$ are integers$)$.

Called an orthogonality relation.

We can use this relation to find the Fourier coefficients.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

Choose an integer q, multiply by $\sin qx$ and integrate.

$$\int_0^{2\pi} f(x) \sin qx \, dx =$$

$$\int_0^{2\pi} \sin(qx) \left[\sum_{n=1}^{\infty} b_n \sin(nx) \right] dx$$

$$= \sum_{n=1}^{\infty} b_n \left(\int_0^{2\pi} \sin qx \sin nx \, dx \right).$$

$$= 0 + 0 + 0 + \ldots + b_q \int_0^{2\pi} \sin^2 qx \, dx$$

$$+ 0 + 0 + 0 + \ldots$$

$$= b_q \pi.$$

So

$$\boxed{b_q = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin qx \, dx}$$

for $q = 1, 2, 3 \ldots$

Similarly, if we can write $f(x)$ as a cosine series.

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx,$$

for later convenience we include a factor of $\frac{1}{2}$.

it turns out that.

$$\boxed{a_q = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos qx \, dx}$$

$q = 0, 1, 2, 3 \ldots\ldots$

because

$$\boxed{\int_0^{2\pi} \cos mx \cos nx \, dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \neq 0. \\ 2\pi & \text{if } m = n = 0 \end{cases}}$$

Also find.

$$\int_0^{2\pi} \sin(mx)\cos(nx)\,dx = 0.$$

m, n integers > 0.

The general case.

Suppose we have a $2L$-periodic function, defined between $x = -L$ and $x = +L$.

Then we can write:

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$