Suppose

Lec 2-3 2010

$$y' = -\frac{y}{x} + \frac{e^{-x^2}}{x}.$$

$$\Rightarrow (xy)' = e^{-x^2}$$

$$xy = \int e^{-x^2}\,dx$$

$$= ?$$

We can't evaluate this integral (i.e express it in terms of simple functions.

Nevertheless the problem is

Solved

$$y = \frac{1}{x}\left[\int_0^x e^{-t^2}\,dt + C\right]$$

arbitrary.

2nd Order Equations

(Still revision) Linear Equations

(1) Constant coefficients.

e.g.

$$y'' + 3y' + 2y = 0.$$

Look for solution $y = \cancel{A}e^{mx}$

$$y' = me^{mx}, \quad y'' = m^2 e^{mx}$$

$$(m^2 + 3m + 2)e^{mx} = 0$$

True for all x if

$$m^2 + 3m + 2 = 0$$ Auxiliary Equation

$$\Rightarrow m = -2 \text{ or } m = -1.$$

So $y = e^{-2x}$ is a solution as is $y = e^{-x}$.

Since the ODE is linear we can add two solutions together to get a third,

General Solution is

$$y = Ae^{-2x} + Be^{-x}$$

arbitrary constants.

If the roots are complex

$$m = p \pm iq.$$

then the general solution is

$$y = e^{px}(A\cos qx + B\sin qx)$$

Why?

$$e^{(p+iq)x} = e^{px} \cdot e^{iqx}$$

$$= e^{px}(\cos qx + i\sin qx)$$

If there is a Right-Hand Side we use the Complementary Function Particular Integral method.

$$y = C.F + P.I.$$

general solution with no RHS

any one solution of full equation.

Method only works if

(a) Equation is Linear

(b) Coefficients are constant
(no x's on LHS).

E.g. $y'' + 3xy' + 2y = 0$

Aaagh!!

How can we solve problems like this?

End of Revision.

New Stuff:

2nd Order Linear ODEs with non-constant coefficients

(1) Method of Variation of Parameters or Reduction of Order.

Begin with example:

Find $y(x)$ satisfying. (†)

$$3x^2 \frac{d^2y}{dx^2} + 5x\frac{dy}{dx} - 5y = 1$$

We instantly spot the P.I.
$y = -1/5$.

Look at $(\dagger)$ with no RHS.
(the "homogeneous" equation)

$$3x^2y'' + 5xy' - 5y = 0$$

Slightly more slowly, we spot that $y = x$ is a solution to this.

$$0 + 5x(1) - 5(x) = 0$$

Having spotted this, return to $(\dagger)$, and substitute

$$y = x\,u(x)$$

Old unknown.

solution we spotted.

New unknown

If $y = xu$

$$y' = (xu)' = 1(u + xu')$$

$$y'' = u' + (xu')'$$
$$= u' + 1.u' + xu''$$

$$y'' = (xu'' + 2u')$$

Substitute in $(\dagger)$

$$3x^2(xu''+2u') + 5x(u+xu') - 5xu = 1$$

Or

$$3x^3u'' + 6x^2u' + 5xu + 5x^2u' - 5xu = 1$$

$$\Rightarrow 3x^3u'' + 11x^2u' + \boxed{0u} = 1$$

The "u" term cancels.

As there is no "u" term, we can write $W = u'$
$W' = u''$

$$3x^3 W' + 11x^2 W = 1$$

A 1st order Linear ODE in $W(x)$.

Which we can solve !!

(using integrating factor).

$$W' + \frac{11}{3x} W = \frac{1}{3x^3}$$

I.F is $e^{\int \frac{11}{3x} dx}$

$$= e^{\frac{11}{3} \ln x}$$

$$= e^{\ln (x^{11/3})}.$$

$$= x^{11/3}.$$

Mult. by I.F.

$$x^{11/3} W' + \frac{11}{3} W x^{8/3} = \frac{1}{3} x^{2/3}$$

$$\frac{d}{dx}\left(x^{11/3} W\right) = \frac{1}{3} x^{2/3}$$

$$x^{11/3} W = \frac{1}{5} x^{5/3} + C.$$

So

$$W = \frac{1}{5}x^{-2} + Cx^{-11/3}$$

$$\parallel$$

$$u'$$

Integrate to find u.

$$u' = \frac{1}{5x^2} + Cx^{-11/3} \cancel{+}$$

$$u = -\frac{1}{5x} + Dx^{-8/3} + E$$

another arbitrary constant

$$\left(-\frac{8}{3}D = C\right),$$

So $y = xu$ gives, finally,

$$y = -\frac{1}{5} + Dx^{-5/3} + Ex$$

(the general solution to (†))

Particular Integral we spotted.

Complementary Function
2 arbitrary constants
2nd Order ODE

Solution to homogeneous equation we spotted.

Why does the method always work?

The general linear Second Order ODE is

$$y'' + a(x)y' + b(x)y = c(x)$$

where $a(x)$, $b(x)$, $c(x)$ are arbitrary functions

[Not always zero]

Suppose we can find any one solution to the homogeneous equation.

$$y'' + a(x)y' + b(x)y = 0.$$

Call it $h(x)$.

Then $h(x)$ satisfies

$$\boxed{h'' + ah' + bh = 0}$$

As before, write

$$y(x) = h(x)u(x)$$

New unknown.

(equivalently, $y = hu$).

$$y = hu$$

$$y' = h'u + hu'$$

$$y'' = h''u + h'u' + h'u' + hu''$$

$$= h''u + 2h'u' + hu''.$$

Sub in ODE.

$$(h''u + 2h'u' + hu'')$$
$$+ a(h'u + hu')$$
$$+ b(hu)$$
$$= C$$

Or

$$hu'' + u'(2h' + ah) + u\underbrace{(h'' + ah' + bh)}_{\text{But this is zero!!}} = C$$

So we have

$$hu'' + u'(2h' + ah) = C.$$

As before, we can write

$$w = u'$$

$$hw' + w(2h' + ah) = C$$

Can be solved by I.F. method.

In summary:

(1) Find solution to homogenous ODE $y = h(x)$.

(2) Substitute $y = hu$.

(3) Get equation $u'' + \{\ \}u' = \{\ \}$

(4) Solve by I.F.

(5). Integrate to find $u \Rightarrow y$.