$u_{\xi\eta} + u_\eta = 0$, write $Z = u_\eta$

$Z_\xi + Z = 0$. $\quad \frac{\partial Z}{\partial \xi} + Z = 0$

$\Rightarrow Z = Ae^{-\xi} = A(\eta)e^{-\xi}$

where A may depend on η!

$u_\eta = A(\eta)e^{-\xi}$

$\Rightarrow u = B(\eta)e^{-\xi} + C(\xi)$

another arbitrary function of η.

Now $\xi = x - y$

$\eta = x - 3y$

So the general solution of our PDE (†) is

$u(x,y) = B(x-3y)e^{y-x} + C(x-y)$

arbitrary function

where B & C are arbitrary functions.

Now on $y=0$, $u = 1 + xe^{-x}$

$u_y = (x-3)e^{-x}$

$\Rightarrow$ On $y=0$: $\underset{\substack{\shortparallel \\ 1+xe^{-x}}}{u(x,0)} = B(x)e^{-x} + C(x)$

$u_y = B'(x-3y)(-3)e^{y-x} + B(x-3y)e^{y-x} + C'(x-y)(-1)$.

On $y=0$

$u_y = (x-3)e^{-x} = -3B'(x)e^{-x} + B(x)e^{-x} - C'(x)$

$$\left.\begin{array}{ll} ① & (x-3)e^{-x} = (B-3B')e^{-x} - C'(x) \\ ② & 1+xe^{-x} = Be^{-x} + C \end{array}\right\}$$ Need to find B & C.

To eliminate C, differentiate ② w.r.t. x and add

② $\Rightarrow (1-x)e^{-x} = (B'-B)e^{-x} + C'$

add to ①

$-2e^{-x} = -2B'e^{-x}$

$\Rightarrow \underline{B'=1} \Rightarrow B = x + \cancel{K}$ constant. w.l.o.g $K=0$,

If we take $K=0$, that merely alters C, so we do

② $\Rightarrow \underline{\underline{C=1}}$ $\quad B(x)=x \Rightarrow B(x-3y) = x-3y$

So we have: $\underline{u(x,y) = (x-3y)e^{y-x} + 1}$

This method works in general. Any equation of the form

$$a u_{xx} + b u_{xy} + c u_{yy} + d u_x + e u_y = f$$

where a, b, c, d, e, f are constants.

Can be reduced by the linear transformation

$$\xi = x + \alpha y, \quad \eta = x + \beta y \quad \text{to}$$

$$u_{\xi\eta} + g u_\xi + h u_\eta = k.$$

called the "canonical" form, provided equation is <u>hyperbolic</u>, <u>i.e. $b^2 > 4ac$</u>.

If equation is parabolic ($b^2 = 4ac$). Write $\xi = x + \alpha y$, $\eta = x$.

$u_x = u_{\xi\xi}$ ($u_\xi = u_{xx}$?).

A simple (yet important) example:

The 1-D wave equation.

$$u_{tt} = c^2 u_{xx}$$

Write $\xi = x + \alpha t$

$\eta = x + \beta t.$

$$u_t = u_\xi \alpha + u_\eta \beta$$

$$u_{tt} = \alpha^2 u_{\xi\xi} + 2\alpha\beta u_{\xi\eta} + \beta^2 u_{\eta\eta}$$

$$u_x = u_\xi . 1 + u_\eta . 1.$$

$$u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}.$$

$$\alpha^2 u_{\xi\xi} + 2\alpha\beta u_{\xi\eta} + \beta^2 u_{\eta\eta} = c^2 (u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta})$$

$$(\alpha^2 - c^2) u_{\xi\xi} + (\beta^2 - c^2) u_{\eta\eta} + (2\alpha\beta - 2c^2) u_{\xi\eta} = 0.$$

Choose $\alpha = c$, $\beta = -c$.

$$-4c^2 u_{\xi\eta} = 0.$$

$$\Rightarrow u_{\xi\eta} = 0$$

$$\Rightarrow u = f(\xi) + g(\eta)$$

where f, g are arbitrary functions

$$u(x, y) = f(x - ct) + g(x + ct)$$

---

wave travelling to the

right

left.

General solution is arbitrary combination.

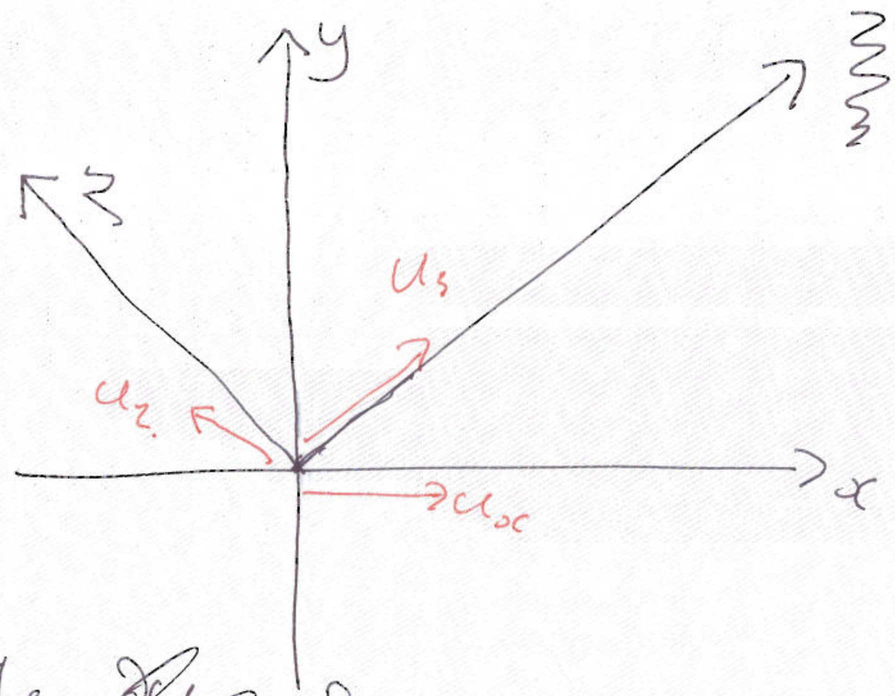
What were we doing when we wrote

$$\xi = ax + by \qquad ?$$
$$\eta = cx + dy$$

$\xi =$ constant is a straight line in (x, y) plane,

ditto $\eta =$ constant.

If $\xi =$ const $\eta =$ const are perpendicular we are essentially rotating axes.



Then ~~$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x}$~~

$$\frac{\partial u}{\partial x} = a\frac{\partial u}{\partial \xi} + c\frac{\partial u}{\partial \eta}$$

(by chain rule).

etc. $\frac{\partial u}{\partial \xi} = \cdots \frac{\partial u}{\partial x} + \cdots \frac{\partial u}{\partial y}$

linear combination

Now

$\frac{\partial u}{\partial x}$ is "the rate of change of u as we move in the x-direction"

Similarly

$\frac{\partial u}{\partial z}$ is "the rate of change of u as we move in the z direction"

and $u_z = \underset{\llcorner\lrcorner}{A}\, u_x + \underset{\llcorner\lrcorner}{B}\, u_y$

constants

$= \begin{pmatrix} A \\ B \end{pmatrix} \underset{\text{dot}}{\cdot} \begin{pmatrix} u_x \\ u_y \end{pmatrix}$

vectors

If $\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ then we are differentiating in the x-direction,

similarly $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ means y-direction

In general, $\begin{pmatrix} A \\ B \end{pmatrix}$ points in the direction we want to differentiate.

Directional Derivatives

Work now in 3-D.

Let $u(x, y, z)$ be a scalar field (i.e a function of position).

~~Then~~ Let $a_1 \underline{i} + a_2 \underline{j} + a_3 \underline{k}$ be a unit vector in same direction.

Then the derivative of u in this direction can be written as:

$(a_1\underline{i} + a_2\underline{j} + a_3\underline{k}) \cdot (u_x\underline{i} + u_y\underline{j} + u_z\underline{k})$

$= a_1 u_x + a_2 u_y + a_3 u_z$

Called the <u>directional derivative</u> of u in the direction of

$\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$.

<u>Notation</u>:

We write

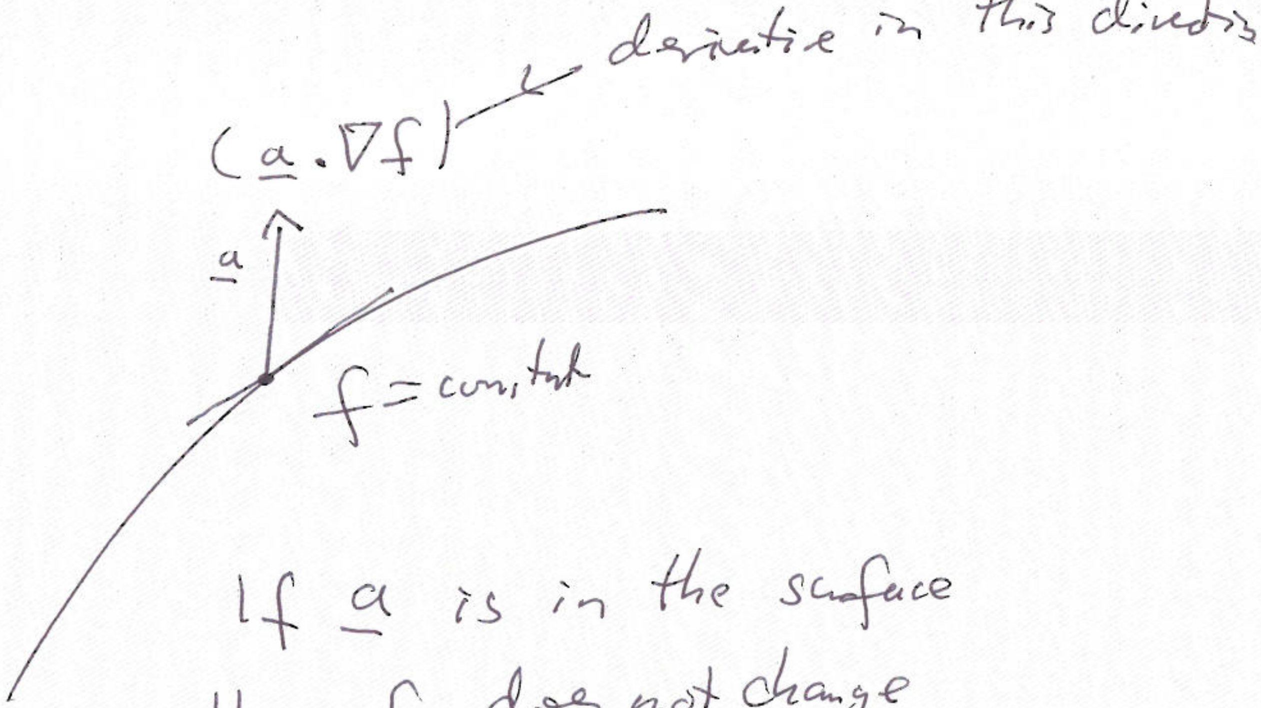
$u_x\underline{i} + u_y\underline{j} + u_z\underline{k} = \nabla u$

"del" or "nabla"

∇ is a "vector operator"

∇u is called "grad u" (short for "gradient") because the vector ∇u points in the direction of maximal increase of u.

Also $|\nabla u|$ is the maximum possible rate of change in u.



If $\underline{a}$ is in the surface
then f does not change
so $\underline{a}.\nabla f = 0$
$\Rightarrow \nabla f$ is $\perp$ to $\underline{a}$
$\Rightarrow \nabla f$ is <u>normal</u> to the surface
$f = \text{constant}$ always and everywhere.

Example (i). $\underbrace{2x + 3y - 5z}_{f} = 14$.

This is a plane.

Normal to $f =$ constant

is $\nabla f = \nabla(2x + 3y - 5z)$

$= \underline{i}\frac{\partial f}{\partial x} + \underline{j}\frac{\partial f}{\partial y} + \underline{k}\frac{\partial f}{\partial z}$

$= 2\underline{i} + 3\underline{j} - 5\underline{k}$

(ii) $x^2 + y^2 + z^2 = 27$.

This is a sphere radius $3\sqrt{3}$ centre the origin.

What is its normal?

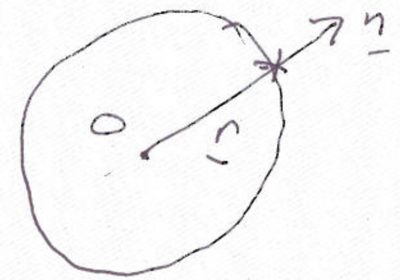
$f = x^2 + y^2 + z^2$

$\nabla f = 2x\underline{i} + 2y\underline{j} + 2z\underline{k}$

$= 2\underline{r}$

$\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$

is the radius vector. ~~to~~ from the origin to the point (x, y, z)



Orthogonal curves

$u = x^2 - 3y^2 = b^2$

$u_x = 2x$

$u_y = -6y$.

$\nabla u = 2x\underline{i} - 6y\underline{j}$

is normal to all those curves