As we want to impose $u=0$ on $y=b$, the last form is convenient,

$$Y = H \sinh w(y-b)$$

So separable solutions are

$$u(x,y) = B_n \sin \frac{n\pi x}{a} \sinh\left[\frac{n\pi}{a}(y-b)\right]$$

So the General solution satisfying the 3 zero b.c.s is

$$u = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right) \cancel{\sinh\left[\frac{n\pi}{a}(y-b)\right]}$$

$$= \sin \pi x$$

$$\Rightarrow B_1 = 1, \quad B_n = 0 \quad n \neq 1.$$

Now impose $u=1$ on $y=0$

$$\Rightarrow 1 = \sum_{n=1}^{\infty} \underline{B_n} \sin\left(\frac{n\pi x}{a}\right) \underline{\sinh\left(-\frac{n\pi b}{a}\right)}$$

for $0<x<a$

Fourier Theory implies

$$C_n \equiv B_n\left(\sinh\left(-\frac{n\pi b}{a}\right)\right) = \frac{2}{a}\int_0^a 1 . \sin\left(\frac{n\pi x}{a}\right) dx$$

$$= \frac{2}{n\pi}\left[-\cos\left(\frac{n\pi x}{a}\right)\right]_0^a$$

$$= \frac{2}{n\pi}\left[1 - \cos(n\pi)\right]$$

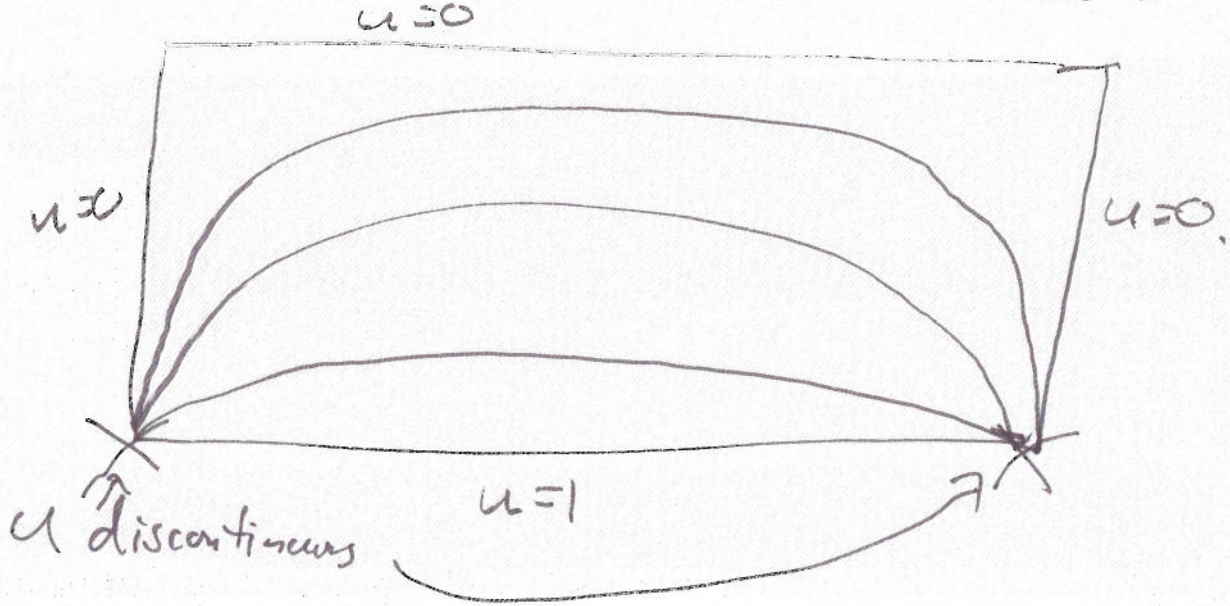
$$= \begin{cases} \frac{4}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

Combining everything, we have

$$u(x,y) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{n\pi} \left[ \frac{\sin\left(\frac{n\pi x}{a}\right) \sinh\left[\frac{n\pi}{a}(y-b)\right]}{\sinh\left(-\frac{n\pi b}{a}\right)} \right]$$

$$= \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi x}{a}\right) \frac{\sinh\left[\frac{n\pi}{a}(b-y)\right]}{\sinh\left(\frac{n\pi b}{a}\right)}$$

Plot contours of $u$



Note suppose, the boundary condition had been

$u = \sin \pi x + \frac{1}{3}\sin(3\pi x)$ at $y = 0$

$\Rightarrow B_n = 0$ for $n \neq 1$ or $3$.

$B_1 = \ldots$ by inspection.

$B_3 = \ldots$

Don't have to use general formula.

Change of Variables —
"Reduction to Canonical Form"

Canonical means "natural or simple or obvious."

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**Example:**

$u(x,y)$ satisfying $(\dagger)$

$$3u_{xx} + 4u_{xy} + u_{yy} + 2u_x + 2u_y = 0$$

With the two initial conditions:

On $y=0$,

$$u = 1 + xe^{-x}$$

$$u_y = (x-3)e^{-x}$$

Try changing variables

$$\xi = x + ay \Rightarrow \xi_x = 1,\ \xi_y = a$$

$$\eta = x + by \Rightarrow \eta_x = 1\ \ \eta_y = b.$$

for constants $a$ & $b$ to be chosen later.

**Chain Rule:**

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$= u_\xi . 1 + u_\eta . 1.$$

$$u_y = u_\xi a + u_\eta b.$$

$u_{xx} = (u_x)_\xi \cdot 1 + (u_x)_\eta \cdot 1$

$= (u_\xi + u_\eta)_\xi + (u_\xi + u_\eta)_\eta$

$= u_{\xi\xi} + u_{\eta\xi} + u_{\xi\eta} + u_{\eta\eta}$

$= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}.$

Similarly,

$u_{yy} = (u_y)_\xi a + (u_y)_\eta b$

$= (au_\xi + bu_\eta)_\xi a + (au_\xi + bu_\eta)_\eta b$

$= a^2 u_{\xi\xi} + 2ab\, u_{\xi\eta} + b^2 u_{\eta\eta}.$

$u_{xy} = (u_x)_\xi a + (u_x)_\eta b$

$= (u_\xi + u_\eta)_\xi a + (u_\xi + u_\eta)_\eta b.$

$= a u_{\xi\xi} + (a+b) u_{\xi\eta} + b u_{\eta\eta}.$

Substitute in (†).

$u_{xx}$

$3(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta})$

$u_{xy}$

$+ 4(a u_{\xi\xi} + (a+b) u_{\xi\eta} + b u_{\eta\eta})$

$u_{yy}$

$+ (a^2 u_{\xi\xi} + 2ab\, u_{\xi\eta} + b^2 u_{\eta\eta})$

$+ 2(u_\xi + u_\eta) + 2(au_\xi + bu_\eta) = 0.$

$u_x$ $u_y$

Gather terms,

$u_{\xi\xi}(3 + 4a + a^2) + u_{\xi\eta}(6 + 4(a+b) + 2ab)$

$u_{\eta\eta}(3 + 4b + b^2) + u_\xi(2 + 2a) + u_\eta(2 + 2b)$

$= 0.$

Chose $a$ and $b$ so that

$3 + 4a + a^2 = 0$

and $3 + 4b + b^2 = 0$, so that $u_{\xi\xi}$ and $u_{\eta\eta}$ terms vanish.

Roots are $-1, -3$.

So ~~wlog~~ without loss of generality take

$a = -1$, $b = -3$. PDE becomes

$$(6 + 4(-4) + 2(3))u_{\xi\eta} + u_\xi(0) + u_\eta(2-6) = 0$$

Or $-4u_{\xi\eta} - 4u_\eta = 0$

Or $u_{\xi\eta} + u_\eta = 0$.

Write $Z = u_\eta$.

$$\frac{dZ}{d\xi} + Z = 0$$