Mixed boundary conditions

Suppose we have

$u = 0$ at $x = 0$

$\dfrac{\partial u}{\partial x} = 0$ at $x = L$.

$$u_t = u_{xx}$$

$$u = X(x)\,T(t)$$

$$\rightarrow \frac{T'}{T} = \frac{X''}{X} = \lambda$$

$$X'' + \lambda X = 0$$

$$\lambda = -\omega^2$$

$X(0) = 0$

$X'(L) = 0$

$$X = A\cos\omega x + B\sin\omega x$$

$X(0) = 0 \Rightarrow 0 = A$

$X = B\sin\omega x \Rightarrow X' = B\omega\cos\omega x$

$X'(L) = B\omega\cos(\omega L) = 0.$

$\Rightarrow \omega L = (n + 1/2)\pi$

(as opposed to $\omega L = n\pi$, previously)

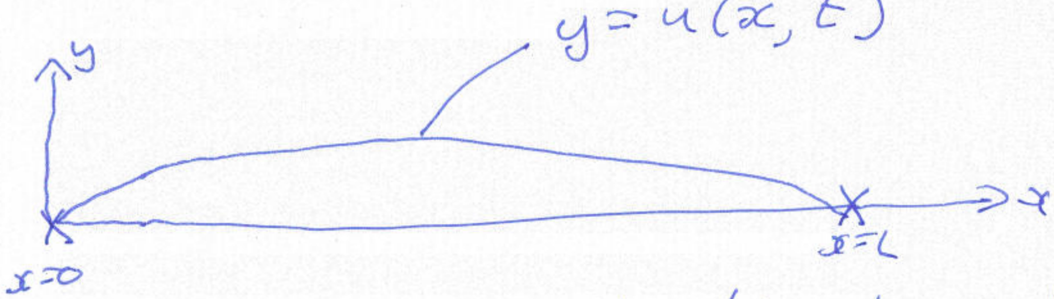
Now continue as before

$$u = \sum_{n=1}^{\infty} B\sin\left[\left(n+\tfrac{1}{2}\right)\frac{\pi x}{L}\right] e^{-\left(n+\frac{1}{2}\right)^2 \frac{\pi^2 t}{L^2}}$$

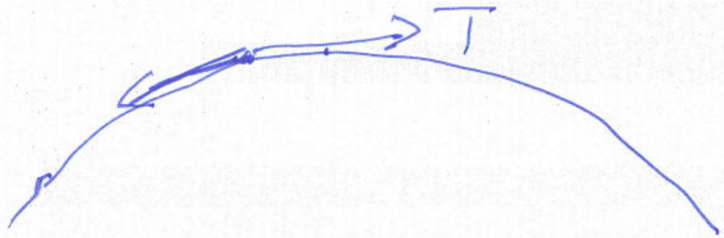
etc.

A Hyperbolic example: The one dimensional wave equation.

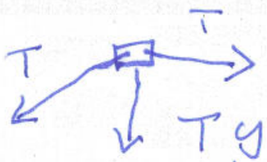
Consider a stretched string, held fixed at both ends



Let the position of the string be $y = u(x,t)$ as in the diagram.



Restoring force is the resultant of the tangential Tension forces in the string



The component perpendicular to the string depends on the curvature of the string, or the rate of change of the gradient along the string

i.e $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$

This balances the "mass × acceleration" of a string portion, i.e

$$\underset{\text{mass density}}{\rho}\ \underset{\text{acceleration}}{\frac{\partial^2 u}{\partial t^2}} = \underbrace{T \frac{\partial^2 u}{\partial x^2}}_{\text{restoring force.}}$$

Write $c^2 = T/\rho$, get 1-D wave eqn: $u(x,t)$ obeys:

$$u_{tt} = c^2 u_{xx}$$

Ends fixed:

$u(0,t) = 0$

$u(L,t) = 0$.

Initial Conditions:

$u(x,0) = f(x)$ initial position

$u_t(x,0) = g(x)$ initial velocity

As before, solve by separation of Variables

Seek solutions $u(x,t) = X(x)T(t)$.

$$\underbrace{XT''}_{u_{tt}} = c^2 \underbrace{X''T}_{u_{xx}}$$

$$\underbrace{\frac{T''}{c^2 T}}_{\text{function of time}} = \underbrace{\frac{X''}{X}}_{\text{function of } x} = \lambda$$

constant.

As before, we want $\lambda < 0$ to get sines/cosines not exponentials.

$\lambda = -k^2$, say

$$X'' + k^2 X = 0$$

$$X = A\cos kx + B\sin kx$$

$$X(0) = 0 \Rightarrow A = 0$$

$$X(L) = 0 \Rightarrow$$

$$B \sin kL = 0.$$

$$\Rightarrow k = \frac{n\pi}{L}, \quad n = 1, 2, 3 \ldots$$

$$X = B \sin\left(\frac{n\pi x}{L}\right).$$

$$\frac{T''}{c^2 T} = \lambda = -\frac{n^2\pi^2}{L^2}$$

$$T'' + \left(\frac{n\pi c}{L}\right)^2 T = 0$$

$$T = \tilde{A}\cos\left(\frac{n\pi c t}{L}\right) + \tilde{B}\sin\left(\frac{n\pi c t}{L}\right)$$

$$u = \left(\tilde{A}B\cos\left(\frac{n\pi c t}{L}\right) + \tilde{B}B\sin\left(\frac{n\pi c t}{L}\right)\right)\sin\left(\frac{n\pi x}{L}\right)$$

is a solution for any n, so ~~sum~~ add them all up to get the general solution

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_n \cos\left(\frac{n\pi c t}{L}\right) + B_n \sin\left(\frac{n\pi c t}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right)$$

What about the initial conditions?

At $t=0$, we want

$$u = f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$$

$\Rightarrow A_n$ known from Fourier Theory,

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)$$

Similarly, we want

$$\frac{\partial u}{\partial t} = g(x) \quad \text{at } t=0$$

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$$g(x) = \sum_{n=1}^{\infty} B_n\left(\frac{n\pi c}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

So we can find B_n from the Fourier Series for g.

$$B_n\left(\frac{n\pi c}{L}\right) = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$B_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Music – What do we hear?

We hear the frequencies $\left(\frac{n\pi c}{L}\right)$ for $n = 1, 2, 3, 4, \ldots$

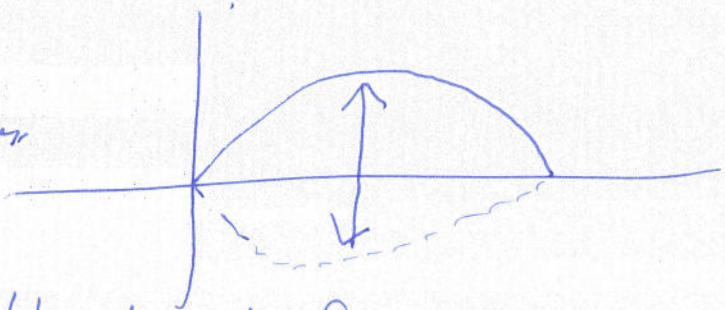
with appropriate amplitudes (A_n, B_n).

General solution is a superposition of all these frequencies

$n=1$

$\sin\frac{\pi x}{L}$

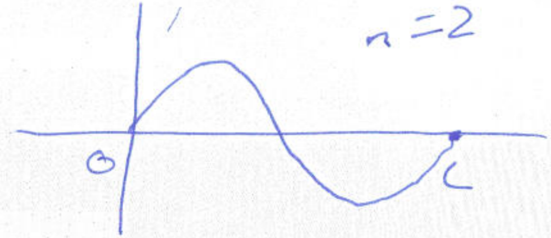
oscillator



with the lowest frequency.

$n=2$ has twice the frequency

An octave higher.



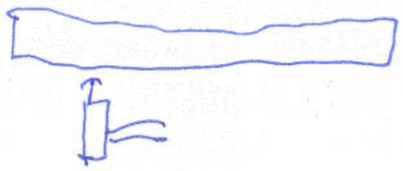
~~$n=3$ has~~

$n=3$ $3 \times$ frequency

$n=4$ $4\times$ frequency two octaves higher.

$n=5$?

$n=6$. octave higher than $n=$?



C E G C is C major chord.

$n=1$ C
2 C
3 G
4 C
5 E
6 G

12 semitones in an octave.

Going up a semitone multiplies frequency by same constant, r, say.

$$\text{So } r^{12} = 2. \qquad \underline{r = 2^{1/12}}$$

Now. $\underline{2^{19} \simeq 3^{12}}$ i.e multiply by 3 is 19 semitones higher.

So $n=3$ is an octave and 7 semitones higher than $n=1$

$n=1$ is C $\Rightarrow$ $n=3$ is G.

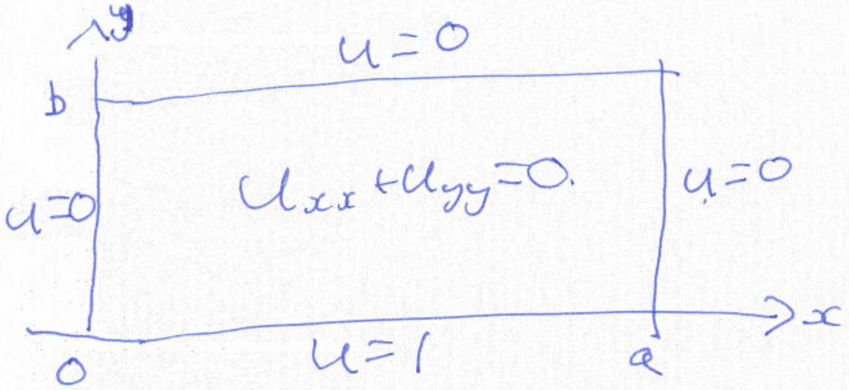
Also $125 \simeq 128$. $\Rightarrow$ $5^3 = 2^7$ $\Rightarrow$ $5 = 2^{7/3} = 2^{\frac{28}{12}}$.

$\simeq$ E two octaves up.

So we can solve the wave equation for any initial conditions.

Elliptic Example:

Laplace's Equation



$u(x,y)$ satisfying

$$u_{xx}+u_{yy}=0$$

in $0<x<a$
$0<y<b$

with say

$u=0$ on $x=0$ and $x=a$

$u=0$ on $y=b$

and $u=1$ on $y=0$

How do we solve this? Use Separation of Variables:

Write $u = X(x)\,Y(y)$.

$$X''Y + XY'' = 0$$

$$\Rightarrow \underbrace{\frac{X''}{X}}_{f \text{ of } x} = \underbrace{-\frac{Y''}{Y}}_{f \text{ of } y.} = \text{constant}$$

Do we choose a positive or a negative constant?

We will either have
exponentials in x
sines in y $\}$

or vice versa.

As the x-boundary conditions have two zeros ($u=0$ at $x=0$ and $x=a$)

Choose negative constant $= -w^2$

$$X'' + w^2 X = 0$$

$$Y'' - w^2 Y = 0$$

So

$$X(x) = A\cos wx + B\sin wx$$

we want $X=0$ at $x=0$ and $x=a$.

$\Rightarrow A=0$ and $\sin(wa)=0$

$\Rightarrow \underline{wa = n\pi}$, $n=1,2,3$ as usual.

$$Y'' = w^2 Y \Rightarrow Y = Ce^{+wy} + De^{-wy}.$$

$$= E\cosh wy + F\sinh wy.$$

$$= G\cosh w(y-b) + H\sinh w(y-b)$$

all these are equivalent.

As we want to impose $u=0$ on $y=b$, the last form is convenient,

$$Y = H \sinh w(y-b)$$

So separable solutions are

$$u(x,y) = B_n \sin \frac{n\pi x}{a} \sinh\left[\frac{n\pi}{a}(y-b)\right]$$

So the General solution satisfying the 3 zero b.c.s is

$$u = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left[\frac{n\pi}{a}(y-b)\right]$$