Last time we solved

$$\boxed{\begin{array}{l} u_t = u_{xx} \\ u = 0 \text{ at } x = 0, 1 \\ u = f(x) \text{ at } t = 0 \end{array}}$$

Suppose instead

$u = T_0$ at $x = 1$ } for all time
$u = 0$ at $x = 0$ }
$u = f(x)$ at $t = 0$ }

How to deal with non-zero boundary condition, $u = T_0$?

Consider the final equilibrium state, when $\frac{\partial}{\partial t} = 0$.

So $0 = u_{xx}$ eventually,

or $u = A + Bx$.

$u = 0$ at $x = 0$

$\Rightarrow A = 0$

$u = T_0$ at $x = 1$

$\Rightarrow \underline{B = T_0}$.

So as $t \to \infty$ we expect

$u \to T_0 x$, a linear profile.

So we write

$u(x,t) = T_0 x + v(x,t)$

new unknown

Then

$u_t = u_{xx} \Longrightarrow$

$v_t = v_{xx}$

$u = 0$ at $x = 0 \Rightarrow v = 0$ at $x = 0$

$u = T_0$ at $x = 1 \Rightarrow v = 0$ at $x = 1$

$u = f(x)$ at $t = 0 \Rightarrow v = f(x) - T_0 x$ at $t = 0$

So

$$\boxed{\begin{array}{ll} v_t = v_{xx} & 0 < x < 1,\ t > 0 \\ v = 0 \text{ at } x = 0, 1. & \\ v = f(x) - T_0 x \text{ at } t = 0 & \end{array}}$$

This is exactly what we solved before.

So can use separation of variables to get

$$v(x,t) = \sum_{n=1}^{\infty} B_n \sin(n\pi x)\, e^{-n^2\pi^2 t}$$

where the constants B_n come from the Fourier Series: at $t = 0$

$$f(x) - T_0 x = \sum_{n=1}^{\infty} B_n \sin(n\pi x)$$

Find B_n, then

$$u(x,t) = T_0 x + \sum_{n=1}^{\infty} B_n \sin(n\pi x)\, e^{-n^2\pi^2 t}$$

$$\left(B_n = 2\int_0^1 (f(x) - T_0 x) \sin(n\pi x)\, dx\right.$$

Insulating Boundary Condition

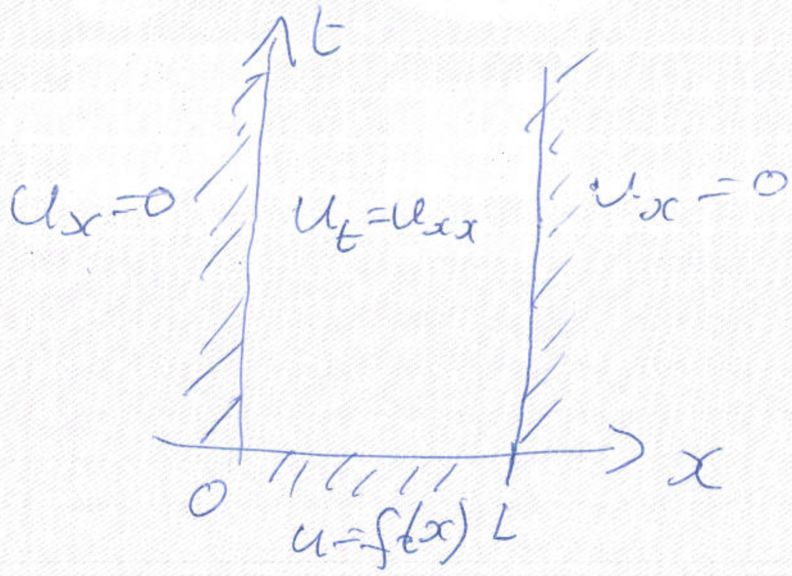
Heat flow is proportional to $(-u_x)$

So for no heat flow at ends we want

$u_x = 0$ at $x=0$ and $x=L$

So try to solve

$$\boxed{\begin{array}{ll} u_t = u_{xx} & 0<x<L \\ & t>0 \\ u_x = 0 \text{ at } x=0 \;\&\; x=L & \\ u = f(x) \text{ at } t=0. & \end{array}}$$



Separate Variables again:

Look for solutions to $u_t = u_{xx}$ of the form $u = X(x)\,T(t)$

$\rightarrow XT' = X''T \Rightarrow \frac{T'}{T} = \frac{X''}{X}$

As before, need $\frac{T'}{T} = \lambda = \frac{X''}{X}$

constant.

As befre, we need $\lambda \leq 0$

Write $\lambda = -w^2$

$$X'' + w^2 X = 0$$

$X = A\cos wx + B\sin wx$

We want $u_x = 0$ at $x = 0, L$

$\Rightarrow X' = 0$ at $x = 0, L$.

$X' = -Aw\sin wx + Bw\cos wx$.

At $x = 0 \Rightarrow \underline{Bw = 0}$.

At $x = L$

$-Aw\sin wL = 0$

This is possible if

either $w = 0$

or $\sin(wL) = 0$.

$\Rightarrow w = \frac{n\pi}{L}$ $n = 0, 1, 2, 3, 4 \ldots$

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and $X = A\cos\left(\frac{n\pi x}{L}\right)$.

And $T' = \lambda T = -\frac{n^2\pi^2}{L^2} T$

$\Rightarrow T = E \exp\left(-\frac{n^2\pi^2}{L^2} t\right)$

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i.e

$u(x,t) = A_n \cos\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2\pi^2}{L^2} t}$

---

is a possible solution for all n and A_n

Adding all these solutions gives the most general solution

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2\pi^2 t}{L^2}}$$

At $t=0$, we want $u=f(x)$.

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$$

Express $f(x)$ as Fourier Cosine Series:

Get $A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$

$n = 1, 2, 3 \ldots$

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx$$

So what happens physically as $t \to \infty$?

$u(x,t) \to A_0$, the average initial temperature.

No heat can escape — just redistributes itself.