(c) Elliptic PDEs

e.g. $\nabla^2\phi = 0$ Laplace's Equation

$\nabla^2\phi = 0$

∇^2 we will meet later

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad \left(+\frac{\partial^2}{\partial z^2} \text{ in 3D}\right)$$

$$\phi_{xx} + \phi_{yy} = 0$$

2-D Laplace equation.

Applications. Myriad. Many Lots.
Multifarious

E.g. Irrotational, Incompressible Fluid Flow $\underset{\sim}{u} = (u, v)$.

$$\left.\begin{aligned} u &= \phi_x \\ v &= \phi_y \end{aligned}\right\} \text{ where } \phi_{xx} + \phi_{yy} = 0.$$

ϕ is the velocity potential

Electrostatics: ϕ could be electric potential, again $\nabla^2\phi = 0$.

Gravity: Mass density ρ.
Gravitational potential ϕ, solves

$$\nabla^2\phi = 4\pi G\rho$$

Poisson's equation (Laplace with a RHS)

Cauchy Riemann Equations (See earlier)

$$u + iv = f(x+iy)$$

$$\left.\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}\right\} \Rightarrow \begin{aligned} u_{xx} + u_{yy} &= 0 \\ v_{xx} + v_{yy} &= 0 \end{aligned}$$

N.B. There is NO time-variable in elliptic problems. Fundamentally different physically.

Classification of 2nd Order PDEs in two variables (Linear)

(See sheet). $u(x,y)$

General 2nd order PDE is

$$a u_{xx} + b u_{xy} + c u_{yy} = f$$

where a, b, c, f are given functions of x & y [and possibly u, u_x, u_y, but not u_{xx}, u_{xy}, u_{yy}]

Theory tells us the sign of $b^2 - 4ac$ is critical.

If

$b^2 - 4ac$
- $> 0 \Rightarrow$ hyperbolic
- $= 0 \Rightarrow$ parabolic
- $< 0 \Rightarrow$ elliptic.

If $b^2 - 4ac > 0$, there exist lines called <u>characteristics</u> along which information can travel (at the wave speed).

In elliptic problems, no such lines exist. There is no <u>time</u> in such problems.

E.g: (a) $u_t = u_{xx}$ $(t \leftrightarrow y)$

$a = 1, b = 0, c = 0$

$b^2 - 4ac = 0. \Rightarrow$ parabolic.

(b) $u_{tt} = u_{xx}$ $(t \leftrightarrow y)$

$a = 1, b = 0, c = -1.$

$b^2 - 4ac = +4 \Rightarrow$ hyperbolic.

(c) $u_{xx} + u_{yy} = 0$

$a = 1,\ c = 1,\ b = 0$

$b^2 - 4ac = -4 < 0$

$\Rightarrow$ elliptic.

(d) $u_{xy} = u_x + u_y + 1.$

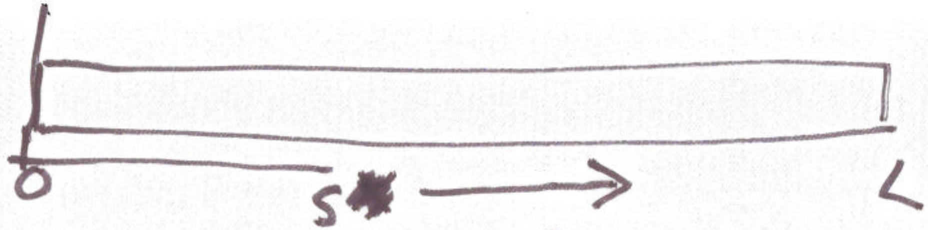
$a = 0,\ b = 1,\ c = 0$

$b^2 - 4ac > 0 \Rightarrow$ Hyperbolic

<u>Solution by Separation of</u>

<u>Variables</u>

Consider heat flow in a bar once more.



Bar in $0 < s < L$

Temperature is

$U(s, \tau)$

τ ← time.

The ends $s = 0$ and $s = L$ we keep at fixed temperature U_0.

At time $\tau = 0$, the temperature distribution is given as

$$U = F(s)$$

where F is a given function.

What is the temperature at later times?

U obeys the heat equation.

$$\frac{\partial U}{\partial \tau} = k \frac{\partial^2 U}{\partial s^2}$$

known thermal diffusivity

i.e

$$\boxed{\begin{array}{ll} U(s,\tau): & \\ U_\tau = k U_{ss} & 0<s<L \\ & \tau>0 \\ U(s,0) = F(s) & \\ U(0,\tau) = U_0 = U(L,\tau) & \end{array}}$$

Non-dimensionalise by defining new variables.

$$x = s/L \qquad t = \frac{\tau \cancel{L^2} k}{\cancel{k}. L^2}$$

$$\frac{\partial U}{\partial \tau} = \frac{\partial U}{\partial t}\frac{dt}{d\tau} = \frac{k}{L^2}\frac{\partial U}{\partial t}$$

$$\frac{\partial^2 U}{\partial s^2} = \frac{1}{L^2}\frac{\partial^2 U}{\partial x^2}$$

$$\frac{k}{L^2}\frac{\partial U}{\partial t} = k.\frac{1}{L^2}\frac{\partial^2 U}{\partial x^2}$$

$$\boxed{\underline{U_t = U_{xx}} \quad \text{in} \quad \begin{array}{l} 0<x<1 \\ 0<t \end{array}}$$

Define $u = U - U_0$

So $u=0$ at $x=0$ and at $x=1$

$$u_t = u_{xx}$$

Also define $f(x) = F(s) - U_0$

So $u = f(x)$ at $t=0$.

giving equation 7 on sheet at last!
(Replace $u_0(x)$ by $f(x)$)

$$\boxed{\begin{aligned} &u_t = u_{xx} \quad 0<x<1,\ t>0 \\ &u = 0 \quad \text{at } x=0 \text{ and } x=1 \\ &u = f(x) \quad \text{at } t=0 \qquad (7). \end{aligned}}$$

Seek special solutions to

$$u_t = u_{xx},$$

called <u>separable solutions</u>, of the form

$$u(x,t) = X(x)\,T(t).$$

where X and T are unknown functions.

Substitute in PDE

$$\frac{\partial u}{\partial t} = X(x)\,T'(t).$$

$$\frac{\partial^2 u}{\partial x^2} = T(t)\,X''(x).$$

so $XT' = TX''$

$$\Rightarrow \underbrace{\frac{T'}{T}}_{\text{a function of } t \text{ only}} = \underbrace{\frac{X''}{X}}_{\text{function of } x \text{ only.}}$$

Only possible if both functions are constant, i.e

$$\frac{T'}{T} = \lambda = \frac{X''}{X} \quad \text{for some } \underset{\text{constant.}}{\lambda}.$$

$\Rightarrow C=0$ (not interesting $C=D=0 \Rightarrow u=0$)

or $m=0$.

But $m=0$

$\Rightarrow X''=0$

$\Rightarrow X=E+Fx$

$\Rightarrow X=0$

if $X(0)=0=X(1)$

So No solutions satisfying

$X(0)=X(1)=0$ if

$\lambda \geq 0$

So take $\lambda<0$

write $\lambda=-\omega^2$

Then

$$X''=-\omega^2 X$$

$$X = C\cos\omega x + D\sin\omega x$$

$$X(0) = C = 0 \qquad X(0)=0$$

$$X(1) = D\sin\omega = 0 \qquad X(1)=0$$

$\Rightarrow D=0$ (not interesting)

OR $\omega = n\pi$, $n=1,2,3\ldots$

In this case $X=D\sin(n\pi x)$

$$T=Ae^{\lambda t}=Ae^{-n^2\pi^2 t}$$

So $u(x,t)=B_n\sin(n\pi x)e^{-n^2\pi^2 t}$

is a solution to $u_t=u_{xx}$

So $T' = \lambda T$

or $T(t) = Ae^{\lambda t}$

and

$X'' = \lambda X$

$\longrightarrow X = e^{\sqrt{\lambda}x}, e^{-\sqrt{\lambda}x}$

if $\lambda > 0$

or $X \sim \cos\sqrt{\lambda}x, \sin\sqrt{\lambda}x$

if $\lambda < 0$.

Physically, we expect we want solutions which decay in time, not grow exponentially (source of energy.) We expect the bar to cool down.

Also, we want $u = 0$ at $x = 0$ <u>and</u> at $x = 1$. This means $X(0) = 0 = X(1)$.

This will be much easier if we have sines/cosines rather than exponentials. So again, probably we want <u>$\lambda < 0$</u>.

If $\lambda > 0$, say $\lambda = m^2$

$X'' = m^2 X$

$\Longrightarrow X = Ce^{mx} + De^{-mx}$

(C, D arbitrary).

Require $X(0) = 0 \Rightarrow C + D = 0$

$X(1) = 0 \Rightarrow Ce^{m} + De^{-m} = 0$

$\Rightarrow C(e^{m} - e^{-m}) = 0.$

$\Rightarrow C(e^{2m} - 1) = 0.$

for any constant B_n and <u>any</u> n integer.

Now since the PDE is <u>linear</u>, if we have any two solutions, we can add them to get a third.

i.e

$$u = \sum_{n=1}^{\infty} B_n \sin(n\pi x) e^{-n^2\pi^2 t}$$

is a solution also for any constants $B_1, B_2, B_3 \ldots$

To complete the solution of (7). we need $u = f(x)$ at $t=0$

i.e we need.

$$f(x) = \sum_{n=1}^{\infty} B_n \sin(n\pi x) \cancel{e^0}$$

This is a Fourier Series! (Half Range, sines only). For any $f(x)$, we can find B_n which gives the solution for all x & t.

Recall the formulae;

$$B_m = 2\int_0^1 f(x)\sin(m\pi x)\,dx$$

---

for $m = 1, 2, 3 \ldots\ldots$

E.g $f(x)=1$.

$$B_n = 2\int_0^1 \sin(n\pi x)\,dx$$

$$= \frac{2}{n\pi}\left[-\cos n\pi x\right]_0^1$$

$$= \frac{2}{n\pi}(1-\cos n\pi).$$

$$= \frac{4}{n\pi} \text{ if } n \text{ is odd.}$$

$$= 0 \text{ if } n \text{ is even}$$

(Square wave example from before).

So the answer is:

$$\boxed{u(x,t) = \sum_{\substack{n=1\\ n\text{ odd}}}^{\infty} \frac{4}{n\pi}\sin(n\pi x)\, e^{-n^2\pi^2 t}}$$

Solution for bar initially at temperature $u=1$, but ends held at temperature $u=0$.

As $t \to \infty$, $u \to 0$ as we expect physically.

For large t, note $e^{-\pi^2 t} \gg e^{-n^2\pi^2 t}$ for $n>1$.

So

$$u \simeq \frac{4}{\pi}\sin(\pi x)\, e^{-\pi^2 t}$$