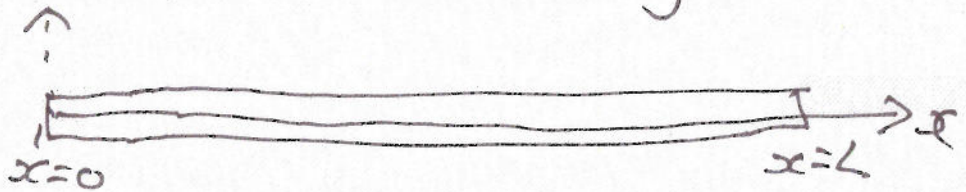
Fourier invented his series so as to be able to solve

The Heat Equation.

Consider a conducting bar



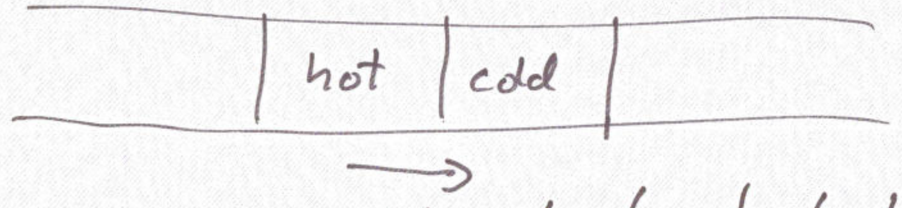
Heat flows from hot to cold

Suppose temperature in bar is $u(x, t)$

position along bar

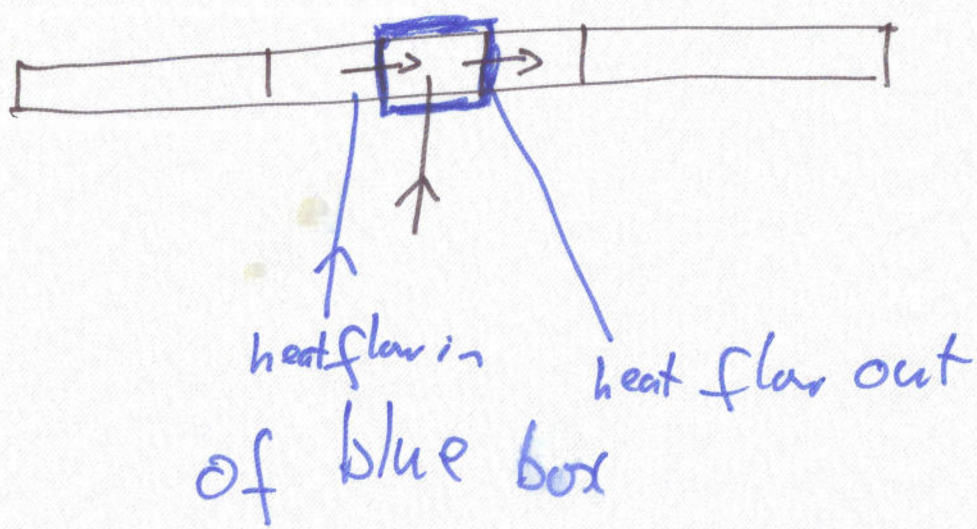
time.

(ignore y & z variation).



Expect heat flow to be proportional to $-\left.\frac{du}{dx}\right|_{\text{time constant}}$ i.e

$$-\frac{\partial u}{\partial x}$$, a partial derivative.



heat in > heat out if $\frac{\partial}{\partial x}(\text{heatflow}) < 0$

Suggests that the equation governing heat flow is:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k \frac{\partial u}{\partial x}\right)$$

assume constant (but could vary with x).

i.e

$$\boxed{\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}}$$

1-D heat equation.

Or $\boxed{u_t = k u_{xx}}$

clearer notation.

This is an example of a 2nd Order [2nd derivative in equation]. Partial Differential Equation (or PDE).

It is also linear.

2nd Order PDES

are very important. They govern the entire known universe.

They are of 3 main types. called parabolic, hyperbolic or elliptic.

(a) Parabolic PDEs

The heat equation is <u>parabolic</u>

$$u_t = u_{xx}$$

also known as the <u>diffusion equation</u> (one-dimensional).

It describes some initial distribution evolving in time towards some smooth equilibrium.

As $t \to \infty$, if $\frac{\partial u}{\partial t} \to 0$

then $u \to A + Bx$

a smooth straight line.

A consequence: We can't run this equation backwards in time: <u>Any</u> initial condition approaches the same end condition. Must be solved forwards in time

(b) Hyperbolic ~~Equations~~ PDEs.

These describe <u>waves</u>

(e.g. electromagnetic radiation,
sound
water waves etc).

Typical example is the 1-D wave equation.

$$u_{tt} = c^2 u_{xx}$$

where c is a constant, the <u>wave-speed</u>.

(Clearly it has dimensions length/time).

[Compare the heat equation

$u_t = k u_{xx}$

where k has dimensions $\frac{(\text{length})^2}{\text{time}}$. k is a <u>diffusivity</u>]

To understand the wave equation

$$u_{tt} = c^2 u_{xx}$$

let's consider the solution

$$u = f(x - ct)$$

where f is any function.

Then

$$\frac{\partial u}{\partial x} = f'(x-ct) \cdot 1 \quad [\text{Chain rule}]$$

$$\frac{\partial^2 u}{\partial x^2} = f''(x-ct) \cdot 1$$

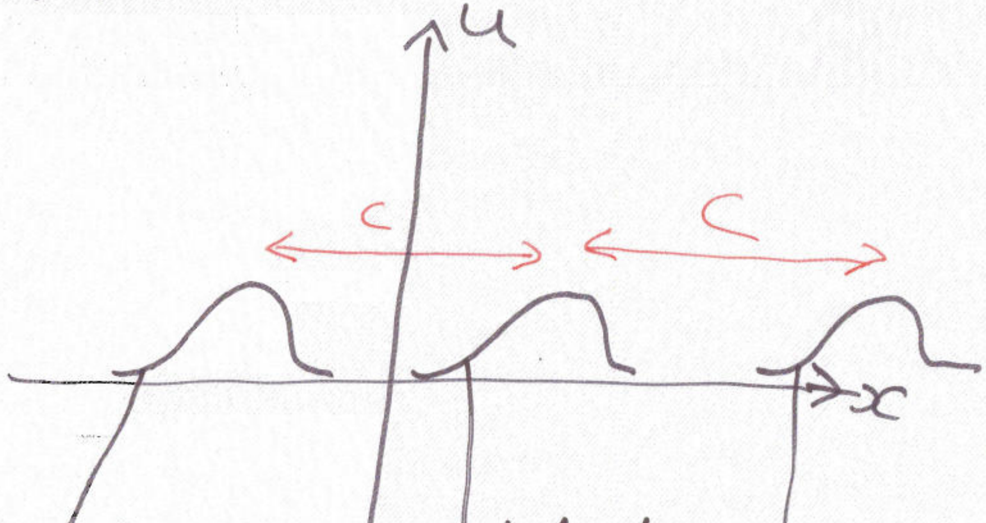
$$\frac{\partial u}{\partial t} = f'(x-ct) \frac{\partial}{\partial t}(x-ct)$$

$$= -c f'(x-ct).$$

$$\frac{\partial^2 u}{\partial t^2} = (-c)^2 f''(x-ct)$$

$$= c^2 \frac{\partial^2 u}{\partial x^2}$$

So $u = f(x-ct)$ is a solution. What does it mean?



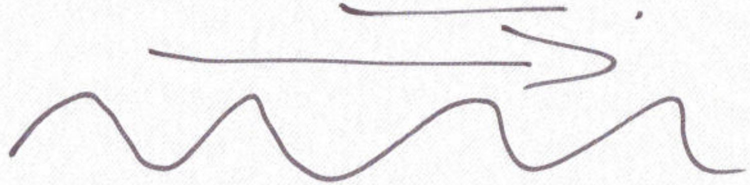
at $t=0$. $u = f(x)$. say it looks like this.

at $t=1$ $u=f(x-c)$

at $t=2$ $u=f(x-2c)$

This solution describes something which moves in the positive x-direction with speed c, without change of shape.

i.e a <u>wave</u>.



$$f(x) = \sin x$$

$$u = \sin(x-ct)$$