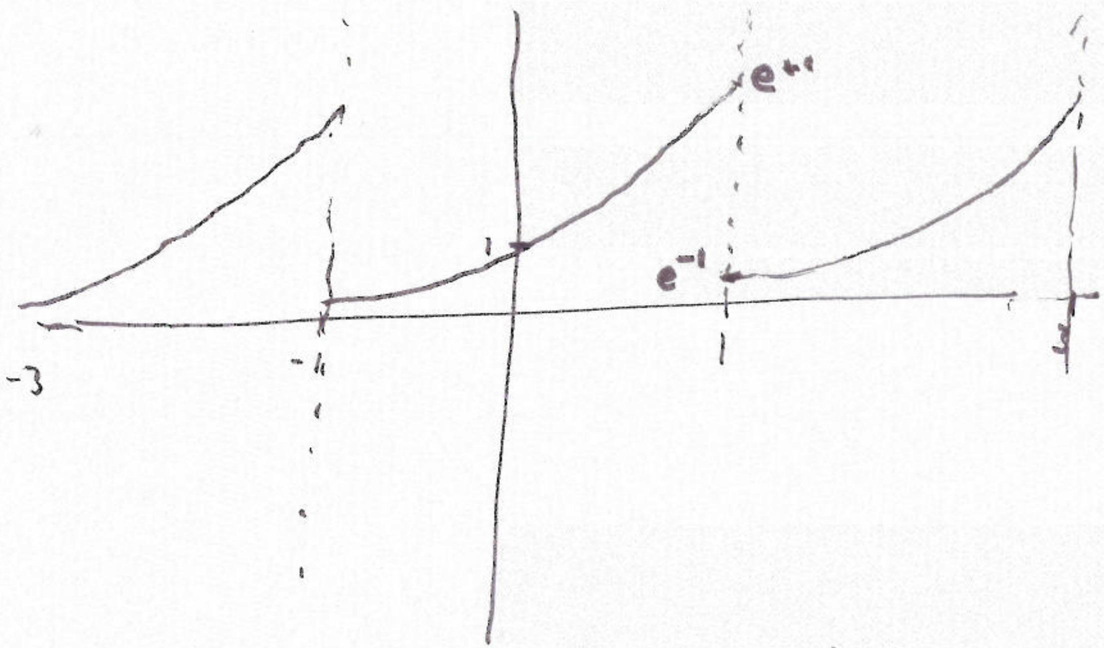
Fourier Series revisited

Example: (Hard)

Let $f(x) = e^x$ for $-1 < x < 1$



and $f(x)$ is 2-periodic as shown. What is its Fourier Series?

In formula for 2L-periodic function take $L=1$.

$$a_n = \frac{1}{1}\int_{-1}^{1} e^x \cos\left(\frac{n\pi x}{1}\right) dx$$

$$b_n = \int_{-1}^{1} e^x \sin(n\pi x)\, dx.$$

Could evaluate these by integrating twice by parts.

Instead consider

$$a_n + i b_n = \int_{-1}^{1} e^x \left(\cos n\pi x + i \sin n\pi x\right) dx$$

$$= \int_{-1}^{1} e^x e^{i(n\pi x)}\, dx$$

$$= \int_{-1}^{1} e^{x(1+in\pi)}\, dx$$

$$= \left[\frac{1}{1+in\pi} e^{x(1+in\pi)}\right]_{-1}^{1}$$

$$= \frac{1}{1+in\pi}\left(e^{1+in\pi} - e^{-1-in\pi}\right)$$

Now

$$e^{1+in\pi} = e^{1}.e^{in\pi}$$

$$= e\left(\cos n\pi + i\sin n\pi\right).$$

$$= e.\cos n\pi$$

$$e^{-1-in\pi} = e^{-1}.e^{-in\pi}$$

$$= e^{-1}.\cos n\pi.$$

So

$$a_n + ib_n = \frac{\cos n\pi\,(e-e^{-1})}{1+in\pi}.$$

$$= \cos n\pi(e-e^{-1})\,\frac{1-in\pi}{1+n^2\pi^2}$$

Equate real and imaginary parts to get:

$$a_n = \frac{\cos n\pi\,(e-e^{-1})}{1+n^2\pi^2}$$

$$b_n = -\frac{n\pi}{1+n^2\pi^2}\,(e-e^{-1})\cos n\pi$$

So

$$e^{x} = (e-e^{-1})\left[\frac{1}{2} + \sum_{n=1}^{\infty} \frac{\cos n\pi}{1+n^2\pi^2}\left(\cos n\pi x - n\pi \sin n\pi x\right)\right]$$

(for $\cancel{x}$ $-1 < x < 1$).

At discontinuity $(x=1)$ series converges to the <u>average</u> of the values either side of the jump i.e $\frac{1}{2}(e^{-1}+e)$.

So

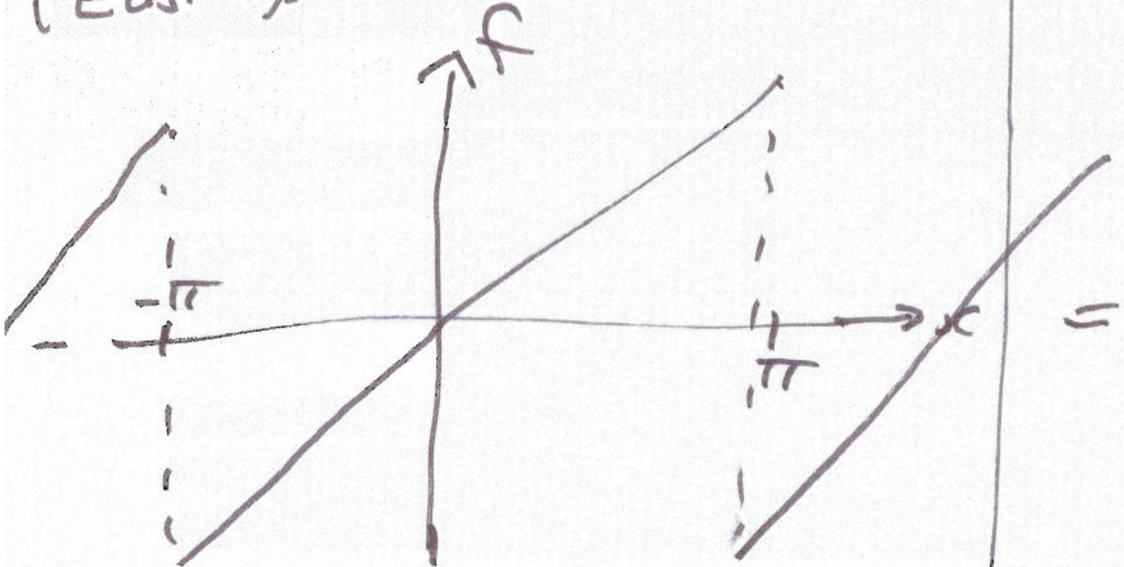
$$\boxed{\frac{1}{2}(e^{-1}+e) = (e-e^{-1})\left[\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{1}{1+n^2\pi^2}\right)\right]}$$

Not at all obvious!

Another example. (2π-periodic)

$f(x) = x$ for $-\pi < x < \pi$

(Easier).



$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cos\left(\frac{n\pi x}{\pi}\right) dx$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin(nx)\, dx$.

As $f(x)$ is odd $a_n = 0$.

for all n.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin(nx)\, dx$$

$$= \frac{1}{\pi}\left(\left[x\frac{(-\cos nx)}{n}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 1 \cdot \left(\frac{-\cos nx}{n}\right) dx\right)$$

$$= \frac{1}{\pi n}\left(-\pi \cos n\pi - \pi \cos(-n\pi)\right) + \int_{-\pi}^{\pi} \frac{\cos nx}{n\pi}\, dx$$

$$= -\frac{2}{n}\cos n\pi + \cancel{\left[\frac{\sin nx}{n^2\pi}\right]_{-\pi}^{\pi}}$$

$\sin n\pi = 0$.

Now $\cos n\pi = (-1)^n$.

$b_n = \frac{2}{n}(-1)^{n+1}$ $\quad n \geq 1$

(N.B. $n=0$ does not apply to b_n).

So

$$x = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1}\sin(nx).$$

(Compare Q2 on problem sheet).

$$f(x) = \frac{1}{2}a_0 + \sum(a_n \cos nx + b_n \sin nx)$$

Any signal

individual "bits" which make up the total

The Fourier coefficients tell us how much of the signal is in each of the Fourier modes $\sin x$, $\cos 3x$ etc.

Parseval's Theorem

essentially says total energy in any signal is equal to the sum of the energies in each mode.

E.g. $f(x)$ 2π-periodic, Total energy (per period) is

$$\frac{1}{\pi}\int_{-\pi}^{\pi} [f(x)]^2\,dx$$

Parseval's theorem states this equals.

$$\frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

$$\cancel{\frac{1}{\pi}\int_{-\pi}^{\pi} f^2\,dx = \frac{1}{2}a_0^2 + \sum_{n=1} a_n^2 + b_n^2}$$

Parseval's Theorem:

In general, ($2L$-periodic f)

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2\,dx = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

E.g for the example

$f(x) = x \quad -\pi < x < \pi$,

$L = \pi$

$a_n = 0$

$b_n = \frac{2}{n}(-1)^{n+1}$.

So we have.

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \sum_{n=1}^{\infty}\left(\frac{2}{n}(-1)^{n+1}\right)^2$$

$$\frac{2}{3}\pi^2 = 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

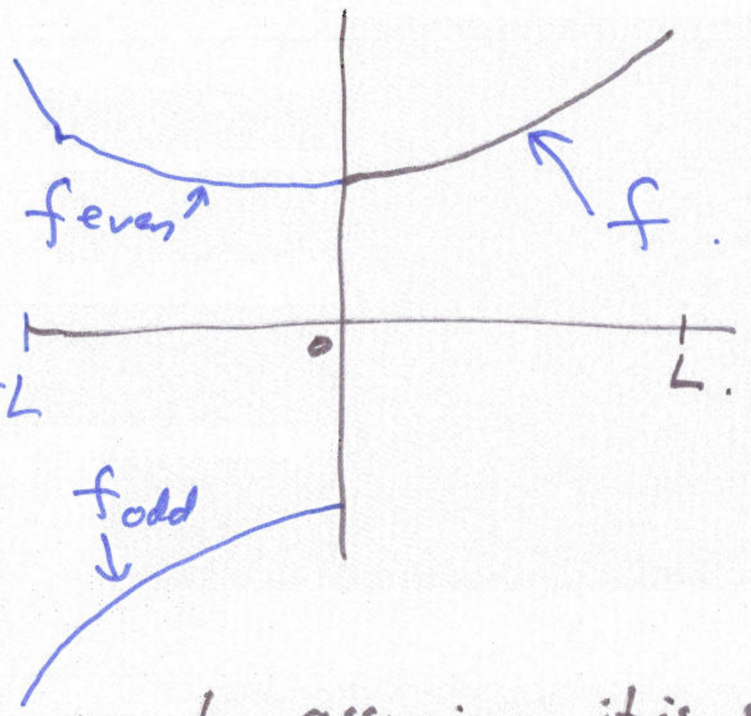
$$= 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots$$

Can derive lots of these surprising formulae this way.

Half-Range Series

Suppose $f(x)$ is defined for $0 < x < L$ only.

If we make $f(x)$ $2L$-periodic,



either by assuming it is <u>even</u> or <u>odd</u>.

we can obtain a F.S. which ~~depends~~ includes only <u>cosines</u> or sines. These are called:

(a) <u>Half-Range Cosine Series (HRCS)</u>

$b_n = 0$ (f assumed even).

$$a_n = \frac{1}{L}\int_{-L}^{L} f_{even}(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$f_{even}(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

(b) Half-Range Sine Series

Now $a_n = 0$

$$b_n = \frac{1}{L}\int_{-L}^{L} f_{odd}(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{2}{L}\int_{0}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$f = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

(H. R. S. S.).

Differentiating a F.S.

Suppose

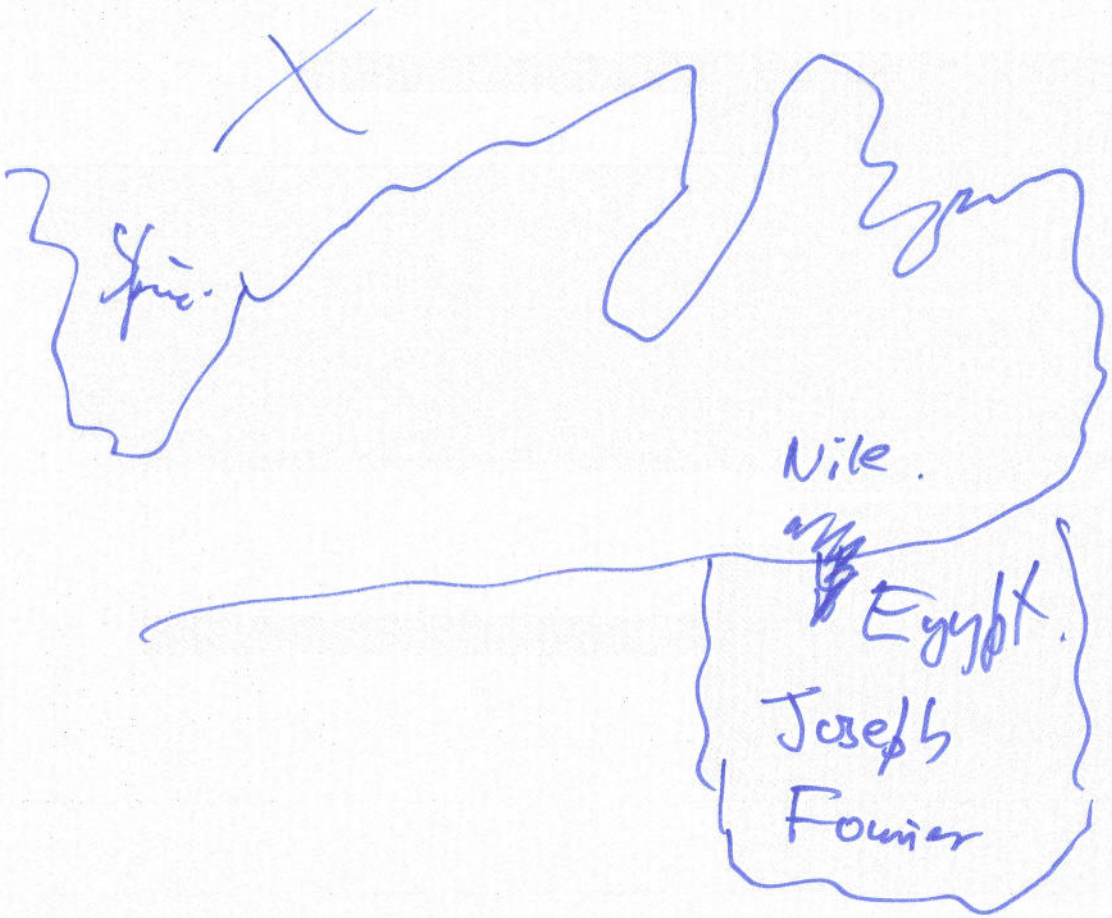
$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

Then we can try differentiating both sides

$$f'(x) = \sum_{n=1}^{\infty} b_n \left(\frac{n\pi}{L}\right) \cos\left(\frac{n\pi x}{L}\right)$$

Another Fourier Series.

Note this n makes the series behave worse. May not converge even. In general, differentiating a signal causes trouble. Wiggles get amplified.


Nile.
Egypt.
Joseph
Fourier