Mathematics

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Books: None in Particular.

Maths for { Scientists / Engineers } G. Stephenson

Syllabus:

- ODEs (Analytic methods only).
- Fourier Series
- P(artial) DES.
- Vector Calculus (grad, div and curl).
- Matrix Algebra. / Complex Variables. / etc } A. Gogolin

Ordinary Differential Equations

(ODEs)

Have an unknown function, say

$y(x)$ ← independent variable

↑ dependent variable

An <u>ODE</u> is an equation involving a derivative of y, e.g

$$\frac{dy}{dx}, \quad \frac{d^2y}{dx^2}, \quad \frac{d^3y}{dx^3} \quad \text{etc.}$$

$$\parallel \qquad \parallel \qquad \parallel$$

$$y' \qquad y'' \qquad y'''$$

equivalently

E.g

$$\frac{dy}{dx} = \sin x + \cos y$$ Order 1

$$y'' = 1 - (y')^2 + (y')^4$$ Order 2

$$y''' + 2y'' + 3y' + 4y = e^{-x}$$ Order 3

The order of an ODE is the number of the highest derivative to appear.

NOTE: $\left(\frac{dy}{dx}\right)^2$ should never be confused with

$$\frac{d^2y}{dx^2} = \left(\frac{d}{dx}\right)^2 y$$

1st Order Equations: You met these last year. We will revisit these briefly.

2nd Order ODEs: You have met constant coefficients, only. We will consider more general cases.

Higher Order ODEs: Reduction to systems of 1st order equations.

Plan

1st Order ODEs: These take the form

$$y' \equiv \frac{dy}{dx} = f(x, y),$$

where f is a given function.

The <u>general solution</u> (the set of all possible solutions) has one arbitrary constant.

[In general, an N'th order ODE has N arbitrary constants in its solution.]

To fix the solution uniquely, we need an extra condition e.g $y(0) = 2$ to find the arbitrary constant.

<u>Revision</u>: What kind of equations can we solve?

(a) $$\frac{dy}{dx} = f(x) \Rightarrow y = \int f(x)\,dx$$

or $$y = \int^x f(t)\,dt + A.$$

(b) <u>Separable ~~solutions~~ equations.</u>

$$\frac{dy}{dx} = f(x,y) = g(x)h(y)$$

$$\Rightarrow \int \frac{dy}{h(y)} = \int g(x)\,dx$$

(c) <u>Linear 1st order ODEs</u>

$$\frac{dy}{dx} + p(x)y = q(x).$$

Can be solved by <u>Integrating factor</u> (I. F.) method.

Example.

$$\frac{dy}{dx} = -\frac{1}{x} y + \sin x.$$

$$y' + \frac{y}{x} = \sin x$$

$$\text{I. F. } e^{\int \frac{1}{x} dx}$$

$$= e^{\log x}$$

$$= x.$$

$$xy' + y = x \sin x$$

$$\frac{d}{dx}(xy) = x \sin x.$$

So

$$xy = \int x \sin x \, dx$$

$$= [x(-\cos x)] - \int 1.(-\cos x) dx$$

$$= -x \cos x + \sin x + C.$$

$$y = -\cos x + \frac{C + \sin x}{x}$$

Suppose we have <u>$y(0)$ is finite</u>.

What is C then?

Must have <u>$C = 0$</u>. ($\lim_{x \to 0} \frac{\sin x}{x} = 1$, See revision sheet).