BioFluids Lecture 15: Steady flow in sightly curved pipes: the Dean equations

Consider flow down a slowly curving pipe. In terms of cylindrical polar coordinates $(r,\ \phi,\ z)$ we shall model this as a portion of a torus, $(r-b)^2+z^2=a^2$ where $b \gg a$, and seek solutions independent of ϕ, driven by a pressure gradient in the ϕ-direction.

The velocity $\mathbf{u}=(u_r,\ u_\phi,\ u_z)$ satisfies the axisymmetric Navier-Stokes equations

$$\left.\begin{aligned} \frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{\partial u_z}{\partial z} &= 0 \\ \rho\left(\frac{Du_r}{Dt}-\frac{u_\phi^2}{r}\right) &= -\frac{\partial p}{\partial r}+\mu\left(\nabla^2 u_r-\frac{u_r}{r^2}\right) \\ \rho\left(\frac{Du_\phi}{Dt}+\frac{u_\phi u_r}{r}\right) &= G(r,z,t)+\mu\left(\nabla^2 u_\phi-\frac{u_\phi}{r^2}\right) \\ \rho\frac{Du_z}{Dt} &= -\frac{\partial p}{\partial z}+\mu\nabla^2 u_z \end{aligned}\right\} \tag{15.1}$$

where the material derivative $D/Dt=\partial/\partial t+u_r\partial/\partial r+u_z\partial/\partial z$. Here $G=-1/r\,\partial p/\partial\phi$ is the downpipe pressure gradient. As we want ∇p to be independent of ϕ, we must have $G=\widehat{G}(t)/r$ only. We write $\widehat{G}=bG_0(1+f(t))$. Let us first see if there is a **unidirectional** solution, as for the straight pipe. If we substitute $u_r=0=u_z$, we find

$$\frac{\partial p}{\partial z}=0 \qquad \text{and} \qquad \frac{\partial p}{\partial r}=\rho\frac{u_\phi^2}{r} \qquad \Longrightarrow \frac{\partial u_\phi}{\partial z}=0\ . \tag{15.2}$$

So such a solution is only possible if u_ϕ is constant on cylinders. Such a flow would be consistent with a no-slip condition only for flows between concentric cylinders. Any curved pipe-flow cannot be unidirectional.

However, if the pipe is almost straight, we might expect the flow to be almost unidirectional. As r and z vary over the scale a, we assume

$$b \gg a \qquad \text{so that} \quad r=b+ax^*\simeq b \qquad \text{and} \quad \frac{\partial}{\partial r}\sim\frac{1}{a}\gg\frac{1}{r}. \tag{15.3}$$

We scale $z=az^*$ and let U_0 be a typical scale of u_ϕ. Then we expect a suitable scale for the pressure to be $p\sim\rho U_0^2 a/b$ and if we scale

$$u_r\frac{\partial u_r}{\partial r}\sim u_z\frac{\partial u_r}{\partial z}\sim\frac{u_\phi^2}{r} \qquad \Longrightarrow \qquad u_r\sim u_z\sim U_0\left(\frac{a}{b}\right)^{\frac{1}{2}}\ . \tag{15.4}$$

Since $b\gg a$ we have, as expected, $u_\phi\gg u_r,\ u_z$. We therefore write

$$u_\phi=U_0u_\phi^* \qquad u_{r,z}=U_0\left(\frac{a}{b}\right)^{\frac{1}{2}}u_{x,z}^* \qquad p=\rho U_0^2\left(\frac{a}{b}\right)p^*, \qquad t=\frac{(ab)^{1/2}}{U_0}t^* \tag{15.5}$$

and neglecting terms of order (a/b), equations (4.1) become

$$\left.\begin{aligned} \frac{\partial u_x^*}{\partial x^*} + \frac{\partial u_z^*}{\partial z^*} &= 0 \\ \frac{\rho U_0^2}{b}\left(\frac{Du_x^*}{Dt^*} - \frac{u_\phi^{*2}}{1}\right) &= -\frac{\rho U_0^2}{b}\frac{\partial p^*}{\partial x^*} + \frac{\mu U_0}{a^2}\left(\frac{a}{b}\right)^{\frac{1}{2}} \nabla^{*2} u_x^* \\ \frac{\rho U_0^2}{(ab)^{1/2}}\frac{Du_\phi^*}{Dt^*} &= G + \frac{\mu U_0}{a^2}\nabla^{*2} u_\phi^* \\ \frac{\rho U_0^2}{b}\left(\frac{Du_z^*}{Dt^*}\right) &= \frac{\rho U_0^2}{b}\frac{\partial p^*}{\partial z^*} + \frac{\mu U_0}{a^2}\left(\frac{a}{b}\right)^{\frac{1}{2}} \nabla^{*2} u_z^* \end{aligned}\right\} \tag{15.6}$$

We choose U_0 to scale with the steady component of pressure gradient and define a parameter K such that

$$\frac{G_0 a^2}{\mu U_0} = 1 \qquad \text{and} \quad K = \frac{\rho U_0 a}{\mu}\left(\frac{a}{b}\right)^{\frac{1}{2}} . \tag{15.7}$$

Dropping the * from all the dimensionless variables we obtain the **Dean** equations:

$$\left.\begin{aligned} \frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} &= 0 \\ K\left(\frac{Du_x}{Dt} - u_\phi^2\right) &= -K\frac{\partial p}{\partial x} + \nabla^2 u_x \\ K\frac{Du_\phi}{Dt} &= 1 + f(t) + \nabla^2 u_\phi \\ K\frac{Du_z}{Dt} &= -K\frac{\partial p}{\partial z} + \nabla^2 u_z \end{aligned}\right\} . \tag{15.8}$$

These equations are essentially the two-dimensional Navier-Stokes equations with a body force u_ϕ^2 acting towards the inside of the bend.

Steady Flow: If we set $f(t) \equiv 0$ and write $\mathbf{u} = (u,\, v,\, w)$ in Cartesian coordinates $(x,\, y,\, z)$, and introduce a stream function, $\psi(x,\, z)$ where $u \equiv u_x = \partial\psi/\partial z$ and $w \equiv u_z = -\partial\psi/\partial x$, and $v(x,\, z) \equiv u_\phi$, then (15.8) reduce to

$$\left.\begin{aligned} K(\mathbf{u}\cdot\nabla v) &\equiv K(\psi_z v_x - \psi_x v_z) = 1 + \nabla^2 v \\ K(\mathbf{u}\cdot\nabla\Omega) &\equiv K(\psi_z \Omega_x - \psi_x \Omega_z) = \nabla^2\Omega - 2Kvv_z \end{aligned}\right\} \tag{15.9}$$

where $\Omega = -\nabla^2\psi$ is the downpipe vorticity and a suffix now denotes a partial derivative. These equations are to be solved for $v(x,z)$ and $\psi(x,z)$ subject to the no-slip conditions

$$\nabla\psi = 0, \qquad v = 0 \qquad \text{on the pipe boundary.} \tag{15.10}$$

There is one parameter in the problem, K, which is known as the Dean number and defined in (15.7). It is a Reynolds number modified by the pipe curvature, (a/b).