BioFluidDynamics: Lecture 13: Pulse propagation in arteries.

See the course Webpage: http://www.ma.ic.ac.uk/∼ajm8/BioFluids

When the heart pressure reaches a certain level the aortic valve opens and about 80cc of blood is injected into the aorta. Artery walls are distensible and an elastic wave is transmitted down the arterial tree. The full problem involves interaction between the 3-D fluid dynamics of the blood and the solid mechanics of wall.

Fortunately, it proves possible to simplify the fluid mechanics when deducing the pressure gradient. Thenafter, elasticity may be neglected in finding the flow details.

1-D model: Assume a straight artery in the x-direction, with cross-sectional area $A(x,t)$ and a mean fluid flow $u(x,t)$. Then with a constant density ρ_0, mass conservation requires

$$A_t + (Au)_x = 0. \tag{13.1}$$

The momentum balance (Euler equation, averaged over A)

$$\rho_0(u_t + uu_x) = -p_x = -\frac{dp}{dA}A_x \qquad \text{assuming } p = p(A). \tag{13.2}$$

To complete the model we need a "Tube-Law," $p = p(A)$, describing the extent to which the artery dilates in response to the excess pressure. We define

$$c^2 = \frac{A}{\rho_0}\frac{dp}{dA} > 0 \qquad \text{for physical sense.} \tag{13.3}$$

Then combining the above, we have

$$\begin{pmatrix} A \\ u \end{pmatrix}_t + \begin{pmatrix} u & A \\ c^2/A & u \end{pmatrix}\begin{pmatrix} A \\ u \end{pmatrix}_x = 0. \tag{13.4}$$

For a circular annular artery $r_0 < r < r_0 + h$, assuming the wall is a thin, uniform Hookean solid with Young's modulus E, we expect

$$E\,\frac{\delta r}{r_0} = \frac{r_0 \delta p}{h}, \tag{13.5}$$

giving a constant value

$$c_0^2 = \frac{A}{\rho_0}\frac{dp}{dA} = \frac{r_0}{2\rho_0}\frac{dp}{dr} = \frac{Eh}{2\rho_0 r_0}. \tag{13.6}$$

This gives a value $c_0 \simeq 5$m/s in the ascending aorta of a young adult human, (arteries stiffen with age). This result agrees well with measurements. Now the observed blood velocity $u \simeq 1m/s$. So $u/c \simeq 0.2$. Furthermore, arterial pressure varies between about 80-120 mm Hg over a heartbeat, so that $\delta p/p \simeq 0.2$ also. If we regard $0.2 \ll 1$, we can linearise (13.1) and (13.2), to obtain

$$A_t + A_0 u_x = 0 \qquad u_t + \frac{c_0^2}{A_0}A_x = 0, \tag{13.7}$$

giving a simple wave equation $c_0^2 A_{tt} = A_{xx}$, with speed c_0. The linear theory predicts that the pulse wave travels down the arterial system without change of shape, but with reflections from bifurcations etc. In fact measurements show the pulse steepens and smoothens.

Nonlinear effects: (13.4) gives the characteristic directions (eigenvalues of the matrix)

$$\frac{dx}{dt} = u \pm c. \tag{13.8}$$

Adding $\pm\, c/A$ times (13.2) to (13.1), we obtain

$$\left[\frac{\partial}{\partial t} + (u \pm c)\frac{\partial}{\partial x}\right] R_{\pm} = 0, \tag{13.9}$$

where

$$R_{\pm} = u \pm \int_{A_0}^{A} \frac{c(A')dA'}{A'} = u \pm \int \frac{dp}{\rho c}. \tag{13.10}$$

The quantities $R_{\pm}$ are Riemann invariants, constant on the appropriate characteristics. Ignoring reflections, this predicts that a simple wave in the positive x-direction, will steepen until a shock forms – like a hydraulic jump. Except for people with an ailment such as a leaky valve, for which u/c is relatively large, the aorta is too short for a shock to form. But steepening of profile well-predicted by inviscid 1-D theory.

Wave intensity: If we denote small changes along neighbouring characteristics by $\delta R_{\pm}$, then we have

$$\delta R_{\pm} = \delta u \pm \frac{\delta p}{\rho c} \qquad \Longrightarrow \qquad \delta p = \tfrac{1}{2}\rho c(\delta R_+ - \delta R_-), \qquad \delta u = \tfrac{1}{2}(\delta R_+ + \delta R_-). \tag{13.11}$$

We then define a quantity I which we call the wave intensity, by

$$\delta I = \delta p\, \delta u = \tfrac{1}{4}\rho c(\delta R_+^2 - \delta R_-^2). \tag{13.12}$$

In this form, forward travelling waves always give a positive contribution to the intensity I, backward ones negative.

Each heart-beat, a forward-travelling wave travels away from the heart until it reaches an arterial bifurcation, where it is partially reflected. "Perfectly matched" branches minimise the reflections, but at a general position and time we expect a mixture of forward and backwards travelling waves. Now R_- is constant on backward characteristics, and it follows that the difference $\delta R_- = 0$ for forward travelling waves. Thus p and u are linearly related near the heart at the start of the cardiac cycle where the wave is almost entirely forward, as illustrated in the figure.