Bio Fluid Mechanics: Lecture 14

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Pulsatile flow in a long straight artery

Steady & Unsteady Reynolds numbers

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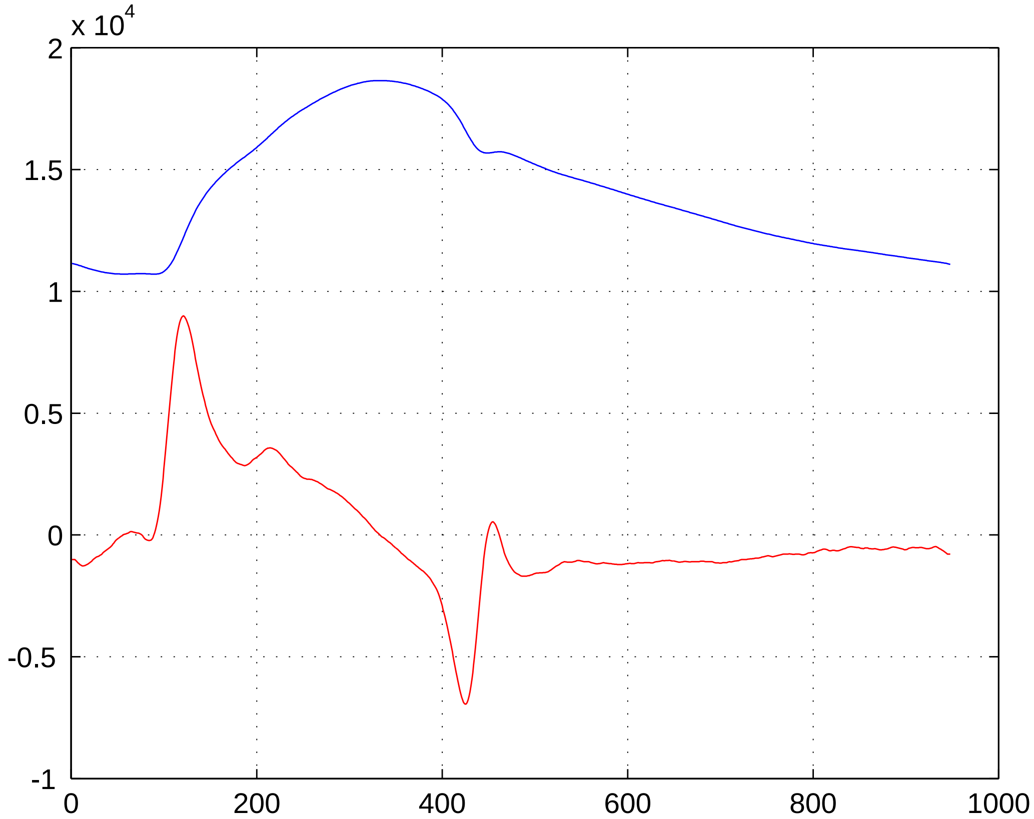
Typical pressure gradient (lower curve) has fairly low mean:



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We can define a steady Reynolds number based on the mean flow rate, and an unsteady Reynolds number based on the peak flow rate. Both these Reynolds numbers are much greater than one in the larger arteries and typically the latter is about 5 times the former. In the human aorta, the peak velocity is about 1m/s, the radius is about 1.5cm and the kinematic viscosity of blood is about 4 times that of water, $\nu = 4 \times 10^{-6} m^2/s$ giving

Peak aortic Reynolds number $\simeq 3750$,   Mean Reynolds number $\simeq 750$

Straight cylindrical model

Today we consider a simple model – assume the artery is a long circular cylinder, $r = a$ in terms of cylindrical polars $(r,\, \theta, z)$. Assume axisymmetric flow.

If the cylinder is long enough for entrance effects to be neglected **[BIG IF]** flow is then one-dimensional and z-independent $\mathbf{u} = (0,\, 0,\, u(r,\, t))$ and governed by the Navier-Stokes equations

$$\rho \frac{\partial u}{\partial t} = G(t) + \mu \nabla^2 u$$

with no slip on the cylinder wall

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Very Important simplification: the geometry is such that $\mathbf{u} \cdot \nabla \mathbf{u} \equiv 0$ so that the problem is linear even at highish Reynolds numbers. Can solve exactly, but danger of unrealistic results.

For a given initial state (u known at $t = 0$) we can find the flow explicitly. Alternatively, and more naturally, we can find the periodic flow which ensues when all initial transients decay.

Decomposition of velocity field

If we similarly decompose

$$u(r,\, t) = \Re e \left[\sum_{n=0}^{\infty} u_n(r) e^{in\omega t} \right]$$

then we can equate harmonics

$$\rho i n \omega u_n = G_n + \mu \nabla^2 u_n$$

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The steady component $(n = 0)$ is simply Poiseuille flow:

$$u_0 = \frac{G_0 a^2}{4\mu} \left(1 - \widehat{r}^2\right) \qquad \text{where} \quad \widehat{r} = r/a$$

The Womersley number

The unsteady $(n > 0)$ parts $u_n(r)$ obey a Bessel-like equation

$$\frac{d^2 u_n}{d\widehat{r}^2} + \frac{1}{\widehat{r}} \frac{du_n}{d\widehat{r}} - in\alpha^2 u_n = -\frac{G_n a^2}{\mu}$$

where the frequency parameter α is given by

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In the aorta, with $a = 0.015$, $2\pi/\omega = 1$ and $\nu = \mu/\rho = 4 \times 10^{-6}$ we have

$$\alpha = 18.8 \qquad \text{in aorta, but smaller in smaller arteries.}$$

From now on we will drop the hat on $\hat{r}$ and regard r as a dimensionless variable.

Unsteady solution

The ODE has a particular solution of a constant. The full solution with the boundary condition can be written

$$u_n = \frac{G_n a^2}{in\mu\alpha^2}\left[1 - \frac{J_0(x\widehat{r})}{J_0(x)}\right] \qquad \text{where} \quad x = i^{3/2}\sqrt{n}\alpha$$

where J_0 is the Bessel function of order zero.
Near $z = 0$, approximately $J_0(z) \simeq 1 - z^2/4$ whereas as $z \to \infty$ $J_0(iz) \simeq e^z/(2\pi z)^{1/2}$ (in a suitable sector of the complex plane).

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Low frequency limit:
As $\alpha \to 0$, the asymptotics give

$$u_n = \frac{G_n a^2}{4\mu}(1 - \widehat{r}^2)$$

the same form as for the steady part. This is the quasistatic limit. The u_t term in the PDE is negligible. As u_n is real, all of the time harmonics are in phase.

High Frequency Limit

High frequency limit

In the aorta the Womersley number $\alpha \simeq 15$ to 20 and so the limit $\alpha \to \infty$ is more appropriate there. Then we have

$$u_n = \frac{G_n a^2}{in\mu\alpha^2}\left(1 - \exp\left[\sqrt{n}\alpha\frac{(1+i)}{\sqrt{2}}(\widehat{r}-1)\right]\right)$$

This is highly reminiscent of the Stokes layer on an oscillating flat plate, or the skin-depth penetration of an alternating magnetic field into an electrical conductor.

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On the surface $\widehat{r} = 1$ we have $u_n = 0$, but the exponential term becomes negligible once $(1-\widehat{r})\alpha$ becomes large. Thus in the main body of the flow, we have u_n is constant, and is independent of the viscosity. The flow simply oscillates in time according to the pressure gradient, and the balance in the PDE is simply

$$\rho\frac{\partial u}{\partial t} = G(t).$$

However, close to the walls, in a layer of thickness $O(\alpha^{-1})$, the velocity adjusts rapidly to zero. Furthermore, the phase of the time oscillation varies quickly with position across the layer. The exponential decay of the higher frequencies is faster.

Time-dependence of Flux

Comparison of the steady and unsteady solutions indicates that

$$\frac{u_1}{u_0} \sim \frac{G_1}{G_0} \frac{4}{\alpha^2}$$

Thus clearly, when $\alpha \gg 1$, a relatively small steady component of the pressure gradient gives rise to a steady velocity which is relatively much larger.

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Integrating the Bessel function analytically, we can calculate the mean velocity

$$\overline{u} = \frac{1}{\pi a^2}\int_0^a 2\pi r u \, dr = \frac{a^2}{\mu}\left[\frac{G_0}{8} + \Re e \sum_{n=1}^{\infty} G_n H(i^{3/2}\sqrt{n}\alpha)e^{ni\omega t}\right]$$

where the function

$$H(x) = \frac{1}{x^2} - \frac{2J_0'(x)}{x^3 J_0(x)}.$$

The following diagram, from McDonald (1974), demonstrates that viscous effects can be important even for the time behaviour of the mean flow rate. McDvisc.jpg

Stress

We can also calculate the wall shear stress,

$$\tau(t) = -\mu \frac{\partial u}{\partial r}\Big|_{r=a} = \frac{G_0 a}{2} - \Re e\left[\sum_{n=1}^{\infty} a G_n \frac{J_0'(x)}{x J_0(x)} e^{ni\omega t}\right] \qquad \text{(writing } x = i^{3/2}\sqrt{n}\alpha \text{ again.)}$$

When α is large this is

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If the steady and peak velocities are of the same order then $G_0 \sim G_n/\alpha^2$. The unsteady part of the stress can therefore be expected to dominate near the wall for large α, if the steady and unsteady velocities are comparable. This also means that the velocity near the wall can reverse.

Here are some animations for different values of α and G_0:

wom1.gif wom2.gif wom3.gif wom4.gif

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- Assumes unidirectional flow
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- No interaction between harmonics
- Predicts spatially uniform wall shear stress
- It does however, give an exact solution, which affords some insight. However, many features are misleading.
- Can be applied also to respiration; periodic flow without a mean component. Indeed, much of what we say about blood flow in arteries also applies to air flow in lungs.

More general high frequency solution.

The crucial simplification in the Womersley solution was the neglect of the nonlinear inertial term $\mathbf{u} \cdot \nabla \mathbf{u}$. In the limit of very high frequency it maybe justifiable to assume

$$\frac{\partial \mathbf{u}}{\partial t} \gg \mathbf{u} \cdot \nabla \mathbf{u}$$

even for more general geometry. Away from the Stokes layers on the walls we can neglect the viscous terms also, giving the problem

$$\nabla \cdot \mathbf{u} = 0, \qquad \rho \frac{\partial \mathbf{u}}{\partial t} = -\nabla p,$$

together with no normal flow on the boundary $\mathbf{u} \cdot \widehat{\mathbf{n}} = 0$. This implies that the pressure p is harmonic ($\nabla^2 p = 0$) and the flow is potential, oscillating in time out of phase with the pressure gradient. Near the walls, we have oscillating Stokes layers, similar to those of the Womersley solution. Predicts wall shear stress with spatial variation given by the slip velocity of the potential flow solution.

This improved solution still neglects important nonlinear interactions and underplays the effects of arterial curvature. We'll begin to look at these next lecture.