

## BioFluids Lectures 7-8: Slender Fish – Added Mass for Lateral Motion

At high Reynolds number, most of the effort required in swimming is pushing water out of the way, that is our energy goes in providing kinetic energy for a sizable volume of fluid. For a thin, streamlined fish this effect for the sideways movements during active swimming is more significant than combatting the inevitable drag due to the net forward motion. To a good approximation the flow can be treated as inviscid, and irrotational outside the wake region.

We consider a fish thin in the  $z$ -direction swimming in the negative  $x$ -direction. From observation it performs small amplitude motions in the  $z$ -direction, and in so doing, propels itself forwards surprisingly fast. Now when it moves its body sideways, it sets some fluid in motion. Thus its lateral movements seem to carry with them a certain amount of ‘added mass’ of water. We begin by formalising this idea.

Consider a circular cylinder of radius  $a$  moving with velocity  $U(t)$  through fluid of density  $\rho$ . If we ignore separation, and treat the flow as potential, then we can write  $\mathbf{u} = \nabla\phi$  where in terms of cylindrical polar coordinates  $(r, \theta, z)$ ,

$$\phi(r, \theta, t) = -a^2 U(t) \frac{\cos \theta}{r}. \quad (7.1)$$

The fluid velocity at the cylinder surface is  $(U \cos \theta, U \sin \theta, 0)$  and slips over the solid boundary which moves with  $(U \cos \theta, -U \sin \theta, 0)$ . and the associated pressure field can be found from the Bernoulli equation

$$p - p_\infty = -\rho \frac{\partial \phi}{\partial t} - \frac{1}{2} \rho |\mathbf{u}|^2 = a^2 \rho \frac{dU}{dt} \frac{\cos \theta}{r} - \frac{1}{2} \rho U^2. \quad (7.2)$$

Now integrating the pressure around the cylinder, we get the force per unit length  $F$ , exerted by the cylinder on the fluid. It is in the  $x$ -direction with magnitude

$$F = \pi a^2 \rho \frac{dU}{dt}. \quad (7.3)$$

Clearly we can regard  $F$  as accelerating a mass per unit length  $m = \pi a^2 \rho$  of fluid. We can also calculate the total kinetic energy of the fluid,

$$E = \frac{1}{2} \rho \int_{r>a} |\nabla\phi|^2 dV = \frac{1}{2} \rho \int_{r=a} \phi \frac{\partial \phi}{\partial n} dS = \frac{1}{2} \rho U^2 \pi a^2 = \frac{1}{2} m U^2, \quad (7.4)$$

which once again suggests an apparent ‘‘added mass’’  $m$  of fluid moving with the cylinder. More generally, we can show that a body of any shape moving with velocity  $\mathbf{U}$  gives rise to fluid kinetic energy

$$E = \frac{1}{2} M_{ij} U_i U_j \quad \text{where } M \text{ is the added mass tensor,} \quad (7.5)$$

and depends only on the instantaneous body shape. To a good approximation the appropriate added mass  $m$  for a given body cross-section can be found from two-dimensional

theory i.e. by finding the added mass for a cylinder of the local cross-section. This approximation requires the wavelength of the body movements to be significantly greater than the cross-section depth,  $s$ . Given this, it is only really the depth  $s$  of the cross-section which is important. Potential flow calculations show that if we write

$$m = \frac{1}{4}\pi s^2 \rho \beta \quad \text{then } \beta \simeq 1 \text{ for most shapes,} \quad (7.6)$$

the relation being exact ( $\beta = 1$ ) for ellipses of any aspect ratio.

### Small amplitude Elongated Body Theory

We now investigate the consequences of Lighthill's 'elongated body theory' for the design of a fish which is to swim efficiently. We have argued that the motion of a slender fish could be well understood by considering the sum of the 'added mass' effects of small amplitude lateral motions. We consider a fish in the  $(x, y)$  plane, with a depth  $s(x)$  in the  $y$ -direction for  $0 < x < L$ . We assume it undergoes displacements  $z = h(x, t)$  as a result of which it swims with a constant velocity  $(-U, 0, 0)$ . To begin with, we work in a frame in which the fish has zero mean motion, so that the fluid at infinity moves past it with uniform speed  $U$ . Then the kinematic boundary condition (equality of the normal components of fluid and solid velocity) requires

$$0 = \frac{D}{Dt}(z - h) \quad \implies \quad w = \frac{Dh}{Dt} \equiv \frac{\partial h}{\partial t} + U \frac{\partial h}{\partial x}, \quad (7.7)$$

where  $w(x, t)$  is the  $z$ -velocity of the fluid at the fish surface. Note that for an inviscid flow this is different from the  $z$ -velocity of the fish, which is

$$W = \frac{\partial h}{\partial t}. \quad (7.8)$$

Then using the local cylinder approximation, the total  $z$ -momentum of the fluid added mass is  $m(x)w(x, t)$ . As this is time-dependent, the fish must exert a force  $F_z$  in the  $z$ -direction, which is

$$F_z = \frac{D(mw)}{Dt}. \quad (7.9)$$

Now this force is provided by the fish, which has to work at a rate  $WF_z$  in order to produce it. Thus the total rate of working of the fish over its entire length is

$$\begin{aligned} P &= \int_0^L WF_z dx = \int_0^L \frac{\partial h}{\partial t} \left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) (mw) dx \\ &= \int_0^L \left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \left( mw \frac{\partial h}{\partial t} \right) dx - \int_0^L mw \frac{\partial w}{\partial t} dx \\ &= \frac{d}{dt} \left[ \int_0^L \left( mw \frac{\partial h}{\partial t} - \frac{1}{2} mw^2 \right) dx \right] + U \left[ mw \frac{\partial h}{\partial t} \right]_0^L. \end{aligned} \quad (7.10)$$

The first term on the RHS of (7.10) clearly has zero time-average for any quasiperiodic motion, and so will not contribute directly to any mechanism enabling steady swimming against a non-zero drag. Now in (7.6) we argued that the added mass  $m(x) = \frac{1}{4}\pi\rho s^2$ , where  $s(x)$  is the total depth of the fish cross-section. In particular, this means that  $m = 0$  at the front of the fish. Many fish have a large caudal fin at their rear, and so  $m$  will be sizable there. Even if their shape tapers to a point in a nicely streamlined shape, we know some separation will occur, and a wake region will commence at some effective non-zero  $s(x)$ . Intriguingly, it appears as though an important factor in swimming is the value of  $mUwW$  at  $x = L$ , and already we might surmise that a large caudal fin might be a good idea. The term  $mUwW|_{x=L}$  can be interpreted as the rate of working required to shed lateral momentum into the wake.

We now derive an alternative expression for the work-rate  $P$ . Comparison with the result (7.10) then enables us to calculate the forwards thrust of the fish, which must balance the drag in the wake, at least on average. Let us now work in a stationary frame, through which the fish swims with speed  $U$ . The sideways velocity components  $w$  and  $W$  are the same as before. Then if the fish exerts a force  $T$  in the swim direction, the rate at which this force does work is  $TU$ . This is included in the total workrate of the fish,  $P$ , along with the rate of increase of kinetic energy of the fluid and the energy flux into the wake. Consistently, we only consider the  $z$ -velocity of the fluid, so that

$$P = TU + \frac{d}{dt} \left[ \int_0^L \frac{1}{2}mw^2 dx \right] + \left[ U\frac{1}{2}mw^2 \right]_0^L. \quad (7.11)$$

Combining the two expressions (7.11) and (7.10), we obtain

$$UT = U \left[ mw(W - \frac{1}{2}w) \right]_{x=L} + \frac{d}{dt} \int_0^L mw \left( \frac{\partial h}{\partial t} - w \right) dx \quad (7.12)$$

so that using (7.7) we obtain the time-dependent thrust  $T$

$$T = \left[ mw(W - \frac{1}{2}w) \right]_{x=L} - \frac{d}{dt} \left[ \int_0^L mw \frac{\partial h}{\partial x} dx \right]. \quad (7.13)$$

Denoting time-averages by  $\bar{\phantom{x}}$ , we obtain the mean thrust and power from (7.13) and (7.11)

$$\bar{T} = \left[ m(\overline{wW} - \frac{1}{2}\overline{w^2}) \right]_{x=L}, \quad \bar{P} = \left[ mU\overline{wW} \right]_{x=L}. \quad (7.14)$$

Now this mean thrust must be positive,  $\bar{T} \geq 0$ , as it must balance the mean drag,  $D$ , which we discussed in lecture 2. It is quadratic in the swim speed  $U$ , and proportional to the cross-sectional area in the direction of swim. This cross-section is increased by lateral movements, so swimming strategies must balance the increased thrust and increased drag obtained by lateral movements. Some fish adopt a swim/glide cycle, accelerating with rapid wiggles, followed by a long flat glide.

### Estimating the efficiency of swimming

We can estimate the efficiency,  $\eta$ , of a swimming cycle by comparing  $U\bar{T}$  and  $\bar{P}$

$$\eta = \frac{\text{Useful work rate}}{\text{Total work rate}} = \frac{U\bar{T}}{\bar{P}} = 1 - \frac{1}{2} \left( \frac{\overline{w^2}}{\overline{wW}} \right)_{x=L}. \quad (7.15)$$

For consistency, we have  $\eta \leq 1$ , so that  $w$  and  $W$  are positively correlated. We thus see that the most efficient swimming occurs when  $w \ll W$  or  $Uh_x \ll h_t$ . However, that leads to small mean thrust and limits the swim speed. Let's take a simple travelling wave as an example. If  $h = -h_0 \sin k(x - Vt - L)$ , then

$$w(L, t) = (V - U)kh_0 \cos kVt \quad \text{and} \quad W(L, t) = kh_0V \cos(kVt). \quad (7.16)$$

Then the mean thrust  $\bar{T}$  and  $\eta$  are given by

$$\bar{T} = \frac{1}{2}m(L)k^2h_0^2(V - U)(V - \frac{1}{2}(V - U)) = \frac{1}{4}m(L)k^2h_0^2(V^2 - U^2), \quad (7.17)$$

and

$$\eta = 1 - \frac{\frac{1}{2}(V - U)^2/2}{V(V - U)/2} = \frac{U + V}{2V}. \quad (7.18)$$

We see that efficient swimming occurs when the swim speed equals the wave speed,  $U = V$ , but this leads to zero thrust and so cannot balance the drag. In practice, we expect the swim speed to be close to, but less than, the wave speed.

### Evolution towards a hydrodynamically perfect fish.

It is clear that conditions at the tail, or caudal fin are vital when swimming. It is therefore no surprise that efficient swimmers have developed a large caudal fin, with resultant large  $m(L)$ . A further evolutionary development has been the change from *anguilliform* to *carangiform* (or *thunniform*) swimming. The anguilliform (eel-like) mode has essentially the entire body undulating, whereas in carangiform mode the travelling wave has sizable amplitude only towards the rear of the body. This has a potential disadvantage known as "recoil". As the caudal fin flaps, it generates a moment about the centre of mass which could cause the fish nose to bend, increasing the drag and cancelling out any advantage. This effect can be minimised by morphological changes, in which the start position of the travelling wave corresponds to a thin cross-section, and very low  $m(x)$  at that point. The high forward mass leads to very limited recoil of the front body. Strictly speaking, we should include a sideways motion and rotation in the theory to date, using a displacement

$$h^*(x, t) = h(x, t) + x\Omega^*(t) + W^*(t), \quad (7.19)$$

rather than the  $h$  we have used so far.  $\Omega^*$  and  $W^*$  should then be determined by Newton's laws for the entire fish. However,  $W^*$  and  $\Omega^*$  are not large in small amplitude motion. The

theory can also be extended to large amplitude displacements without too much effort, but we won't investigate that here.

Observations of a variety of fish shapes and behaviour indicate two changes as the swimming mode becomes more carangiform. Simultaneously, there is a reduction in extent of the dorsal fin, which is much used in anguilliform motion, a decrease in the minimum cross-section forward of the caudal fin and thirdly a thickening of the front portion of the body. All these changes can be understood in terms of the above theory.

A further evolutionary change is the development of the "lunate tail." The shedding of vorticity into the wake takes place at the tips of the caudal fin; there is no particular advantage in having a continuous tail, which can increase the drag a little. Some of the fastest swimmers have a caudal fin which resembles swept-back wings. Some of the preceding theory requires mild modification for such designs.

So what is the caudal fin doing in these fast swimmers such as the tuna? As it waggles it sheds vortices into the wake, from alternate sides, which at first sight look like a Karman vortex street. However, the sense of each vortex is opposite from that of the street, so that there is a rearward flux of momentum in the wake.

**Schooling:** Schools of fish can use the wake structure to improve their swimming efficiency, in a manner reminiscent of 'slip-streaming'. While it would be a mistake for two fish to swim one behind each other, net benefit can derive from a diamond shape pattern, as in the figure. Other advantages of schooling relate to interactions with predators.

Another way of understanding what is actually happening in the wake of one of these fast swimming fish is to regard the caudal fin as acting like a wing, shedding vorticity in order to generate circulation and a resultant lift force. As the tail oscillates, this force always has a forward component – the fish, quite literally, flies through the water.

This suggests we should begin to consider lift generation and animal flight... which we shall do next time.

*"There be four things which I know not:*

*The way of an eagle in the air; the way of a serpent upon a rock; the way of a fish in the midst of the sea; and the way of a man with a maid."*

**Proverbs 30:18-19.**