Typical Bio-Fluid Dynamics exam questions

1. Steady flow in a curved artery of circular cross-section is modelled by axisymmetric flow around a large torus. Discuss the advantages and limitations of such a model.

In terms of cylindrical polar cppoordniates $(r,\ \phi,\ z)$, the steady axisymmetric Navier-Stokes equations for the velocity $\mathbf{u} = (u_r,\ u_\phi,\ u_z)$, are

$$\left.\begin{aligned} \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{\partial u_z}{\partial z} &= 0 \\ \rho\left(\mathbf{u}\cdot\nabla u_r - \frac{u_\phi^2}{r}\right) &= -\frac{\partial p}{\partial r} + \mu\left(\nabla^2 u_r - \frac{u_r}{r^2}\right) \\ \rho\left(\mathbf{u}\cdot\nabla u_\phi + \frac{u_\phi u_r}{r}\right) &= G + \mu\left(\nabla^2 u_\phi - \frac{u_\phi}{r^2}\right) \\ \rho\mathbf{u}\cdot\nabla u_z &= -\frac{\partial p}{\partial z} + \mu\nabla^2 u_z \end{aligned}\right\} \qquad (1)$$

where G is the downpipe pressure gradient, $G = -1/r\, \partial p/\partial\phi$.

Explaining any approximations and scalings you make, derive the Dean equations

$$\left.\begin{aligned} K(\psi_z v_x - \psi_x v_z) &= 1 + \nabla^2 v \\ K(\psi_z \Omega_x - \psi_x \Omega_z) &= \nabla^2\Omega - 2Kvv_z \end{aligned}\right\} \qquad (2)$$

where $\Omega = -\nabla^2\psi$ and suffices now denote partial derivatives. Define the non-dimensional parameter K

These equations are to be solved subject to no-slip on the boundary $r = 1$. If $K \ll 1$, obtain the leading order solutions for v and ψ in terms of polar coordinates.

2. Give a brief outline of the physical origins of 'Profile Drag' and 'Induced Drag' in steady flight of a rigid body.

In steady horizontal flight, a bird travelling at speed U relative to the air exerts a mean thrust T and has to work at a rate $P = TU$ where for suitable positive constants A and B

$$T = AU^2 + \frac{B}{U^2}.$$

Sketch $P(U)$ roughly.

When a bird migrates with constant speed U, it wishes to maximise the distance travelled for a given expenditure of energy. Explain why the tangent from the origin in the (U, P)-plane touches the curve at the optimal migration speed through stationary air (ignoring stability considerations). Calculate this speed in terms of A and B.

Now suppose the air is moving relative to the ground with a uniform speed V in the direction of travel. What function should the migrating bird aim to maximise?

Show how this can be found from your graph. Is this optimum speed (relative to the air) greater or less than the optimum migration speed in still air?

3. Consider the following two-dimensional model of blood flow:

Flow is driven in the rigid channel $a > y > -a$ by the oscillating pressure gradient

$$-\frac{\partial p}{\partial x} = G_0 + G_1 \cos \Omega t \ ,$$

where G_0, G_1 and Ω are constants with $G_1 \gg G_0 > 0$.
Seek a unidirectional solution to the incompressible Navier-Stokes equations of the form $\mathbf{u} = (u(y,t),\, 0,\, 0)$ with

$$u = u_0(y) + \Re e \left[u_1(y) e^{i\Omega t} \right] \ ,$$

where $\Re e$ denotes the real part. Find u_0 and show that for a suitable real constant δ,

$$u_1 = \frac{G_1}{\rho i \Omega} \left[1 - \frac{\cosh[(1+i)y/\delta]}{\cosh[(1+i)a/\delta]} \right] \ .$$

As $\Omega \to \infty$, show that the wall shear stress

$$\mu \left. \frac{\partial u}{\partial y} \right|_{y=a} \quad \to \quad -aG_0 - G_1 \left(\frac{\mu}{2\rho\Omega} \right)^{1/2} \left[\cos \Omega t + \sin \Omega t \right] \ .$$

Discuss whether or not the velocity can be negative for some values of y and t, and comment on the implications for blood flow.

4. Describe the assumptions behind "Resistive Force Theory" and explain how it enables the propulsive force of a flagellum undergoing prescribed movement to be calculated.

A long thin needle-shaped organism of length l moves with speed U through fluid of viscosity μ at low Reynolds number. The drag is $2k\mu l U$ if it moves perpendicular to its axis and $k\mu l U$ if it moves parallel to its axis, where k is a dimensionless constant.

The needle is inclined at an angle α to the vertical and falls under gravity. Use the linearity of the Stokes equations to show that its velocity makes an angle β to the vertical where

$$\tan(\alpha - \beta) = \tfrac{1}{2} \tan \alpha \ .$$

Show that the maximum possible value of β is given by $\tan \beta = 1/(2\sqrt{2})$ and that this occurs when $\tan \alpha = \sqrt{2}$. For this maximal value of β show that the speed of the organism is

$$U = \frac{mg}{\sqrt{2} k \mu l}.$$

$$\left[\text{Recall that} \quad \cos x = \frac{1}{(1 + \tan^2 x)^{1/2}} \quad \text{and} \quad \tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y} \ . \right]$$

5. Explain when and why gyrotaxis occurs. What is its qualitative effect if a uniform distribution of upwards-swimming organisms is placed in a Poiseuille flow in a vertical cylinder?

Bioconvection of a uniform concentration of bacteria is governed by the equations

$$\left.\begin{aligned} \nabla\cdot\mathbf{u} &= 0 \\ \frac{\partial\mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u} &= -\nabla P+\alpha gc\widehat{\mathbf{z}}+\nu\nabla^2\mathbf{u} \\ \frac{\partial c}{\partial t}+\mathbf{u}\cdot\nabla c &= -\nabla\cdot(cV\widehat{\mathbf{p}})+D\nabla^2 c. \\ \widehat{\mathbf{z}}\wedge\widehat{\mathbf{p}} &= B\nabla\wedge\mathbf{u} \end{aligned}\right\} \tag{3}$$

Explain the significance of all variables and terms in each equation. If the uniform state $c = c_0$, $V = V_0$ $\widehat{\mathbf{p}} = \widehat{\mathbf{z}}$ is perturbed with perturbations proportional to

$$\exp(ikx + ily + st) \qquad \text{with} \quad \kappa^2 = k^2 + l^2,$$

it is found that the dispersion relation is

$$(s + D\kappa^2)(s + \nu\kappa^2) = \beta\kappa^2 \qquad \text{where} \quad \beta = \alpha g c_0 V_0 B.$$

Determine which values of κ give rise to instability, and show that maximum growth rate is

$$s_{max} = \frac{\beta}{(\sqrt{D}+\sqrt{\nu})^2}.$$

6. Write an essay on any problem considered in the course. You should include the biological background, the assumptions and approximations made in the model and the solution methods,as appropriate.

The above questions are intended to be roughly of exam standard, but it is conceivable that some may be too easy or too hard. They vary from the complete bookwork (i.e. essentially covered completely in lectures) to the tangential.

We agreed last term that we would hold an examples class some time in the first few weeks of term. I assume you would still like this – we should get in touch by e-mail to fix a time.

Hope you had a good Easter break,

Jonathan Mestel