A NUMERICAL APPROACH TO SPECTRAL RISK MEASURES

ANTOINE JACQUIER

Abstract. We focus here on some very recent results obtained by Cherny and Madan (see [2], [3], [4], [5], [6], [7]) who developed a rigorous mathematical framework for the study of coherent risk measures. Our main contribution is to provide some numerical and empirical facts concerning spectral risk measures, with a particular emphasis on coherent acceptability indices proposed in [4]. One result, for instance, is that this index is infinite for distributions that are symmetric around 0.

1. Introduction to spectral risk measures on L^∞

In the following, we will consider a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For a random variable X with continuous distribution, we denote by $q_\lambda(X) = \inf\{x : \mathbb{P}(X \leq x) > \lambda\}$ for $\lambda \in [0,1]$ the right λ-quantile of X.

1.1. **Convex and coherent measure of risk.** We might sometimes speak of risk measures, sometimes of utility functions. If ρ is a risk measure, then the corresponding utility function u reads $u(X) := -\rho(X)$.

Definition 1.1. A mapping $\rho : L^\infty \to \mathbb{R}$ is a convex risk measure if and only if the corresponding utility u satisfies:

- Monotonicity: If $X \leq Y$ then $u(X) \leq u(Y)$
- Translation invariance: $\forall m \in \mathbb{R},\ u(X+m) = u(X) + m$
- Concavity: $\forall \lambda \in [0,1], u(\lambda X + (1-\lambda)Y) \geq \lambda u(X) + (1-\lambda) u(Y)$

Definition 1.2. A coherent utility function on L^∞ is a concave utility function with the additional positive homogeneity property:

$$\forall \lambda \geq 0,\ u(\lambda X) = \lambda u(X).$$

One can easily see that, for coherent risk measures, the convexity property reduces to a *superadditivity property*. Furthermore, any coherent risk measure is normalised, i.e. $\rho(0) = 0$. Dealing with utility functions is more convenient than with coherent risk measures as it allows us to get rid of many minus signs. In the following, we denote $\mathcal{P}$ the set of probability measures on $\mathcal{F}$ that are absolutely continuous with respect to the reference probability $\mathbb{P}$. We have the following representation theorem (see [14], Proposition 4.14 p. 161):

Proposition 1.3. *A function u is a coherent utility function on L^∞ if and only if there exists a nonempty set $\mathcal{D} \subseteq \mathcal{P}$ such that*

$$u(X) = \inf_{\mathbb{Q} \in \mathcal{D}} \mathbb{E}_{\mathbb{Q}}(X).$$

The author acknowledges the support received from the European Science Foundation (ESF) for the activity 'Advanced Mathematical Methods for Finance'. He would like to thank Alexander Cherny from Moscow State University and Monique Jeanblanc from Evry University for their detailed comments and corrections, Claude Martini for his constant support as well as all the Mathematics Department in Evry for providing him the opportunity to work on this topic.

We note that the set $\mathcal{D}$ may obviously not be unique. We therefore define the determining set as the largest L^1-closed convex subset of $\mathcal{P}$ satisfying the above theorem (see [5]). Note that we work here on L^∞. This representation theorem can be extended to $L^p, p \geq 1$ with some additional conditions on the risk measure. We now provide a few examples.

1.2. **Weighted VaR.** We first define the tail Value-at-Risk. Let $\lambda \in [0,1]$, then the TVaR ([14], p.180) of order λ is defined as

$$u_\lambda (X) := \inf_{\mathbb{Q}\in\mathcal{D}_\lambda} \mathbb{E}_{\mathbb{Q}} (X), \quad \text{where } \mathcal{D}_\lambda = \left\{\mathbb{Q} \in \mathcal{P} : d\mathbb{Q}/d\mathbb{P} \leq \lambda^{-1}\right\}.$$

Let μ a probability measure on $[0,1]$. The weighted VaR utility function is then defined by

$$u_\mu (X) := \int_{[0,1]} u_\lambda (X) \mu (d\lambda),$$

and, from Proposition 1.3, we have the representation

$$u_\mu (X) = \inf_{\mathbb{Q}\in\mathcal{D}_\mu} \mathbb{E}_{\mathbb{Q}} (X),$$

where $\mathcal{D}_\mu$ is the determining set of u_μ. We refer to [5] for the representation of this set. The following theorem will be of fundamental importance for us:

Theorem 1.4. *Let u_μ be a weighted VaR utility function, then there exists an increasing, concave function Ψ_μ on $[0,1]$ with $\Psi(0) = 0$, $\Psi_\mu(0+) = \mu(\{0\})$ and $\Psi_\mu(1) = 1$ such that*

$$u_\mu (X) = \int_{\mathbb{R}} x d\left(\Psi_\mu (F_X (x))\right),$$

where F_X is the distribution function of X. The function Ψ is called a distortion function.

To prove it, we need the following lemma:

Lemma 1.5. *Let μ be a probability measure on $[0,1]$. Define a function $\Psi : [0,1] \to \mathbb{R}$ as $\Psi(0) = \mu(\{0\})$ and*

$$\Psi (x) := \mu (\{0\}) + \int_0^x \int_{(t,1]} s^{-1} \mu (\mathrm{d}s)\, \mathrm{d}t \quad \text{for all } 0 < x \leq 1.$$

Then Ψ is increasing, concave and $\Psi(1) = 1$. Furthermore, $\Psi'(x) = \int_x^1 \mu(\mathrm{d}s)/s$.

Proof. The fact that Ψ is increasing is obvious. Now, let $\lambda \in [0,1]$ and $(x,y) \in (0,1]^2$. We have

$$\Psi (\lambda x + (1-\lambda) y) - \mu (\{0\}) = \int_0^{\lambda x + (1-\lambda)y} \int_{(t,1]} \frac{\mu(\mathrm{d}s)}{s} \mathrm{d}t = \int_0^{\lambda x} \int_{(t,1]} \frac{\mu(\mathrm{d}s)}{s} \mathrm{d}t + \int_{\lambda x}^{\lambda x + (1-\lambda)y} \int_{(t,1]} \frac{\mu(\mathrm{d}s)}{s} \mathrm{d}t.$$

For the first integral, we have

$$\int_0^{\lambda x} \int_{(t,1]} s^{-1} \mu (\mathrm{d}s)\, \mathrm{d}t = \lambda \int_0^x \int_{(\lambda t,1]} s^{-1} \mu (\mathrm{d}s)\, \mathrm{d}t,$$

and hence

$$\lambda \int_0^x \int_{(\lambda t,1]} s^{-1} \mu (\mathrm{d}s)\, \mathrm{d}t + \lambda \mu (\{0\}) \geq \lambda \Psi (x). \tag{1.1}$$

Now,

$$\int_{\lambda x}^{\lambda x+(1-\lambda)y} \int_{(t,1]} s^{-1}\mu(\mathrm{d}s)\,\mathrm{d}t = (1-\lambda)\int_0^y \int_{(\lambda x+(1-\lambda)t,1]} s^{-1}\mu(\mathrm{d}s)\,\mathrm{d}t,$$

and

$$(1-\lambda)\int_0^y \int_{(\lambda x+(1-\lambda)t,1]} s^{-1}\mu(\mathrm{d}s)\,\mathrm{d}t + (1-\lambda)\mu(\{0\}) \geq (1-\lambda)\Psi(y)\,. \tag{1.2}$$

Combining (1.1) and (1.2) proves the concavity of Ψ. Moreover

$$\begin{aligned}\Psi(1) &= \mu(\{0\}) + \int_0^1 \int_{(t,1]} s^{-1}\mu(\mathrm{d}s)\,\mathrm{d}t \overset{\text{Fubini}}{=} \mu(\{0\}) + \int_{(t,1]} s^{-1}\int_0^1 \mathbb{I}_{\{t<s\leq 1\}}\mathrm{d}t\mu(\mathrm{d}s)\\ &= \mu(\{0\}) + \mu((0,1]) = 1\end{aligned}$$

□

The following theorem now holds

Theorem 1.6. *Let $X \in L^\infty$ and q_X be the quantile function of X, then*

$$\rho_\mu(-X) = \Psi(0+)\operatorname{ess\,sup}(X) + \int_{(0,1]} q_X(t)\,\Psi'(1-t)\,\mathrm{d}t = \int_{-\infty}^0 [\Psi(P(X>x))-1]\,\mathrm{d}x + \int_0^{+\infty}\Psi(P(X>x))\,\mathrm{d}x.$$

Proof. We recall the following facts, for $\lambda \in [0,1]$:

$$V@R_\lambda(-X) = \inf\{\alpha : P(-X\leq -\alpha)\leq\lambda\} = \inf\{\alpha : P(X\leq\alpha)\geq 1-\lambda\} = q_X(1-\lambda)\,,$$

and the average V@R is defined as $AV@R_\lambda(X) := \frac{1}{\lambda}\int_0^\lambda V@R_\gamma(X)\,d\gamma$. Now,

$$\begin{aligned}\rho_\mu(-X) &= \int_{(0,1]} AV@R_\lambda(-X)\,\mu(d\lambda) = \int_{(0,1]}\frac{1}{\lambda}\int_0^\lambda V@R_\gamma(-X)\,d\gamma\mu(d\lambda)\\ &= \int_{(0,1]} q_X(1-\gamma)\,d\gamma\int_\gamma^1\frac{\mu(d\lambda)}{\lambda} = \Psi(0+)\operatorname{ess\,sup}(X) + \int_{(0,1]} q_X(t)\,\Psi'(1-t)\,\mathrm{d}t.\end{aligned}$$

This proves the first equality of the theorem. Now, for the second one, first assume that X only takes positive values. Then

$$\begin{aligned}\int_0^1 q_X(t)\,\Psi'(1-t)\,\mathrm{d}t &= \int_0^1 \Psi'(1-t)\int_0^{+\infty}\mathbb{I}_{\{F_X(x)<t\}}\mathrm{d}x\mathrm{d}t\\ &\overset{Fubini}{=} \int_0^{+\infty}\mathrm{d}x\int_0^1\Psi'(1-t)\,\mathbb{I}_{\{F_X(x)<t\}}\mathrm{d}t = \int_0^{+\infty}\mathrm{d}x\int_0^1\Psi'(t)\,\mathbb{I}_{\{t<1-F_X(x)\}}\mathrm{d}t\\ &= \int_0^{+\infty}\mathrm{d}x\int_0^{1-F_X(x)}\Psi'(t)\,\mathrm{d}t = \int_0^{+\infty}[\Psi(1-F_X(x))-\Psi(0+)]\,\mathrm{d}x\\ &= \int_0^{+\infty}\Psi(1-F_X(x))\,\mathrm{d}x - \Psi(0+)\operatorname{ess\,sup}(X)\,,\end{aligned}$$

where we used the fact that $q_X(t) = \sup\{\alpha : P(X\leq\alpha)<t\} = \int_0^{+\infty}\mathbb{I}_{\{F_X(x)<t\}}\mathrm{d}x$. In this case $(X\geq 0)$, we remark that the first integral in the second equality of the theorem is null, indeed, for all $x\leq 0$, $P(X>x)=1$ implies $\Psi(P(X>x)-1)=0$. For a general $X\in L^\infty$, we let $C := -\operatorname{ess\,inf}(X)$, i.e $X+C\geq 0$. By translation invariance, $\rho_\mu(-(X+C)) = \rho_\mu(-X)+C$, and the theorem follows. □

We can now prove Theorem 1.4:

Proof. If we now integrate by part the second equality of the above theorem, we have, for $X \in L^{\infty}$:

$$
\begin{aligned}
\rho_{\mu}(-X) &= \int_{-\infty}^{0} [\Psi(P(X > x)) - 1]\, \mathrm{d}x + \int_{0}^{+\infty} \Psi(P(X > x))\, \mathrm{d}x \\
&= [x\Psi(-F_X(x))]_{-\infty}^{0} + [x\Psi(P(X > x))]_{0}^{+\infty} - \int_{-\infty}^{0} xd(\Psi(-F_X(x))) - \int_{0}^{+\infty} xd(\Psi(P(X > x))) \\
&= -\int_{-\infty}^{0} xd(\Psi(-F_X(x))) - \int_{0}^{+\infty} xd(\Psi(-F_X(x))) \\
&= -\int_{-\infty}^{+\infty} xd(\Psi(-F_X(x))) = \int_{\mathbb{R}} xd(\Psi \circ F_X)(x),
\end{aligned}
$$

where we assume $\lim_{x\to-\infty} x\Psi(-F_X(x)) = \lim_{x\to+\infty} x\Psi(P(X > x)) = 0$, so that the two brackets are null. □

2. Extension to convolution semigroup

We here extend the notion of weighted VaR introduced above following [9].

2.1. Coherent acceptability indices.

Definition 2.1. (see [2])[1] A map $\alpha : L^{\infty} \to \mathbb{R}_+$ is a coherent acceptability index if and only if there exists a collection $(\mathcal{D}_x)_{x\in\mathbb{R}_+}$ of subsets of $\mathcal{P}$ such that for $x \le y$, $\mathcal{D}_x \subseteq \mathcal{D}_y$ and

$$\alpha(X) = \inf\left\{x \in \mathbb{R}_+ : \inf_{\mathbb{Q}\in\mathcal{D}_x} \mathbb{E}_{\mathbb{Q}}(X) < 0\right\} \text{ with } \inf\emptyset = \infty.$$

This definition also reads

$$\alpha(X) = \inf\left\{x \in \mathbb{R}_+ : \mathbb{E}_{\left(\Psi_{\mu}^{x}\circ F_X\right)\mathbb{P}}(X) < 0\right\} \text{ with } \inf\emptyset = \infty,$$

where the family $\left(\Psi_{\mu}^{x}\right)_{x\in\mathbb{R}_+}$ is defined relatively to the family of determining sets $(\mathcal{D}_x)_{x\in\mathbb{R}_+}$. This holds true because we saw before that for a given WV@R, the infimum was attained on the determining set of the risk measure. Furthermore, Cherny and Filipovic [9] recently narrowed the class of distortion functions and introduced a so-called convolution semigroup:

Theorem 2.2. *Let μ be a probability measure on $[0,1]$. A family $\left(\Psi_{\mu}^{t}\right)_{t\ge0}$ of distortions is a concave distortion semigroup is and only if there exists a concave function $G : [0,1] \to \mathbb{R}_+^*$ such that*

$$\forall t \ge 0, \forall x \in (0,1],\ \Psi_{\mu}^{t}(x) = \inf\left\{y \in [x,1] : \int_x^y G(s)^{-1}\, \mathrm{d}s = t\right\},$$

where $\inf\emptyset = 1$. *Furthermore we have the inverse relation*

$$\forall x \in (0,1),\ G(x) = \lim_{t\searrow0} \left(\Psi_{\mu}^{t}(x) - x\right)/t.$$

[1] For clarity and brevity reasons, Cherny and Madan (see their revised version of [8]) now prefer to use the term acceptability index, dropping the coherent term.

2.2. **Statement of the problem.** To summarise the above notations, there is a one-to-one correspondence between a spectral risk measure and a convolution semigroup of increasing concave functions. We would like to study the class of spectral risk measures more deeply. Starting from a historical probability $\mathbb{P}$, we can define the coherent probability measure $\mathbb{Q}_\mu^x$ associated to the coherent acceptability index defined above as

$$\forall x \in \mathbb{R}_+, \ \frac{d\mathbb{Q}_\mu^x}{d\mathbb{P}} = \partial \Psi_\mu^x \circ F_X, \tag{2.1}$$

which is directly linked with the pricing via utility functions that, in the traditional framework of complete markets reads $d\mathbb{Q}/d\mathbb{P} = cU'(X)$ where $\mathbb{P}$ is the physical measure, $\mathbb{Q}$ the risk-neutral one, U the utility function of an agent and c a normalising constant. In an incomplete market framework, there might be an infinity of equivalent martingale measures such that (2.1) is satisfied. Our problem can be decomposed into the following subproblems:

- We might not be able to observe the whole semigroup, but just one element of the family (think, for instance, about rating transition matrices: we just observe the 1-year matrix, but in order to price a 6-month Credit-default-swaps, we need the 6-month matrix). So, which conditions must we impose on a function Ψ so that the family is indeed a distortion convolution semigroup ? How can we reconstruct the whole semigroup from one element of the family? Is there unicity? For the rating transition matrix analogy, some results have already been obtained by [15].
- What are the consequences on the measure $\mathbb{Q}_\mu^x$ and its Radon-Nikodym derivative with respect to $\mathbb{P}$?

3. Application and numerical results

In this section, we wish to provide some numerical results. Basically, what we do is the following: consider some distributions (preferably fat-tailed distributions, though we will also provide the corresponding results for the Gaussian), and numerically and graphically study the risk associated (with respect to the different families of risk measures); this will also provide us with the corresponding coherent acceptability index (whenever it exists). Let X be a random cashflow over a given period of time and $(\Psi_t)_{t\geq 0}$ a distortion function. Then the risk measure is given by

$$\rho_{\Psi_t}(X) = \int_{\mathbb{R}} q_{-X}(u)\, \Psi_t'(1-u)\, du.$$

In the following, whenever there is no ambiguity on the distortion function we are using, we will write ρ_t instead of ρ_{Ψ_t}. Consider a portfolio with 0 value at the beginning of the period, so that the total loss of the portfolio is exactly $L = -X$. As in [7], we defined an acceptability index $\alpha(X)$ as follows: we are interested here in finding the optimal index t^* such that

$$t^* = \alpha(X) = \sup\{t \geq 0 : \rho_t(X) < 0\},$$

with $\sup\{\emptyset\} = \infty$ For practical reasons, we take here a strict inequality for the definition of the index.

3.1. **Results: overview and methodology.** The methodology is as follows: We discretise the integral corresponding to the spectral risk measure (50 steps); we numerically (using the quantile function in MAPLE) invert the probability distribution of the simulated Loss process. For consistency reasons, for a fixed ν, we keep the same simulated loss distribution for all t and all x. When the product $q \cdot \Psi$ is converging sufficiently fast to 0 in 0 and 1, then MAPLE is able to obtain some results in a reasonable period of time. For more rigorous results, in particular for the Gaussian or the Gumbel distributions, it is much more efficient to implement the numerical integration in a fast-computing environment, such as C++, or Python. We used Python here. We refer the reader to the appendix concerning the distortion used here. In particular, $\Psi^{(1)}$ refers to the AIW index, $\Psi^{(2)}$ to the AIMIN index, $\Psi^{(3)}$ to the AIMAX index, $\Psi^{(4)}$ to the exponential utility function. Numerically, if we want to compute such a risk measure, we bump into several problems. The most immediate one is to determine the quantile function of the distribution, which we very scarcely have in closed-form. Suppose that we use a software providing the quantile function at each point (such as MAPLE for instance), if we want to discretise the integral, we have a further problem around 0: the quantile function goes to $-\infty$ whereas $\Psi'(.)$ tends to $+\infty$. So we need a discretisation step too large to allow stable results. Another approach, which we take here, is to consider an approximate analytical formula for the quantile function and derive an analytical result for the integral. We have

$$\forall u \in \mathbb{R},\ \operatorname{erf}(x) = 2\pi^{-1/2}\int_0^x \mathrm{e}^{-u^2}du,\ F_{X_{0,1}}(x) = (2\pi)^{-1/2}\int_{-\infty}^x \mathrm{e}^{-u^2/2}du = 1/2 + 1/2\operatorname{erf}\left(x/\sqrt{2}\right),$$

where $X_{0,\sigma^2} \hookrightarrow \mathcal{N}\left(0,\sigma^2\right)$. Hence

$$\forall x \in [0,1],\ F^{\leftarrow}_{X_{0,1}}(x) = \sqrt{2}\operatorname{erf}^{-1}(2x-1) = \left(F^{\leftarrow}_{X_{\mu,\sigma^2}}(x) - \mu\right)/\sigma,$$

so that $q_{-X_{0,1}} = \left(q_{-X_{\mu,\sigma}} + \mu\right)/\sigma$. And we deduce the spectral risk measure associated to the Gaussian distribution:

$$\rho_{\Psi_t}\left(X_{\mu,\sigma}\right) = \int_0^1 \left(\sigma q_{-X_{0,1}}(u) - \mu\right)\Psi_t^{\prime(i)}(1-u)\,du = -\mu + \sigma\sqrt{2}\int_0^1 \operatorname{erf}^{-1}(2u-1)\,\Psi_t'(1-u)\,du$$

for all $t \geq 0$. Numerically speaking, we could either use a brute Taylor series expansion (see appendix) for the erf^{-1} function, but some more efficient methods have been proposed, in particular, the Acklam algorithm to compute the inverse cdf of the Gaussian. The following theorem is very general:

Theorem 3.1. *Let X be a distribution symmetric around 0 and $\left(\Psi_t(.)\right)_{t\geq 0}$ a concave distortion semigroup such that, $\forall x \in [0,1]$, $\partial_x \Psi_0(x)$ is a constant (which is the case for the AIMIN and AIMAX distortions), then*

$$\forall t > 0,\ \rho_{\Psi_t}(X) > 0,\ \rho_{\Psi_0}(X) = 0,\ \text{and } \alpha(X) = \infty.$$

Proof.

$$\begin{aligned}
\forall t > 0,\ \rho_{\Psi_t}(X) &= \int_0^1 q_{-X}(u)\,\Psi_t'(1-u)\,du \\
&= \int_0^{1/2} q_{-X}(u)\,\Psi_t'(1-u)\,du + \int_{1/2}^1 q_{-X}(u)\,\Psi_t'(1-u)\,du \\
&= \int_0^{1/2} q_{-X}(u)\,\Psi_t'(1-u)\,du + \int_0^{1/2} q_{-X}\left(u+\frac{1}{2}\right)\Psi_t'\left(\frac{1}{2}-u\right)du \\
&= \int_0^{1/2} q_{-X}(u)\left[\Psi_t'(1-u) - \Psi_t'\left(\frac{1}{2}-u\right)\right]du,
\end{aligned}$$

where we used the fact that $q_{-X}(u+1/2) = -q_{-X}(u)$ for $u \in [0,1/2]$. Now, for $u \in [0,1/2]$, we have $(1-u) \in [1/2,1]$, $(1/2-u) \in [0,1/2]$, and because, for each $t > 0$, $\Psi_t(.)$ is a concave function,

$\Psi_t'(1/2-u) > \Psi_t'(1-u)$, for $u \in [0,1/2]$. We also have that, for $u \in [0,1/2[,\ q_{-X}(u) < 0$, and hence the result. The second statement of the theorem is immediate form the last line of the formula and the last statement is an immediate consequence of the first statement. □

3.2. **Analytical results for the Gaussian distribution.** We saw above that, for the $\mathcal{N}\left(0,\sigma^2\right)$ distribution, we always have $\rho_{\Psi_0^{(2)}}(X) = \rho_{\Psi_0^{(3)}}(X) = 0$ because the distribution is symmetric around 0. Suppose now that $X_\mu \hookrightarrow \mathcal{N}\left(\mu,\sigma^2\right)$, with $\mu \neq 0$, then we have the trivial identities:

$$F_{-X_{\mu,\sigma}}(x) = F_{-X_{0,1}}((x+\mu)/\sigma), \quad \text{and } q_{-X_{\mu,\sigma}}(x) = \inf\left\{\alpha : F_{-X_{\mu,\sigma}}(\alpha) \geq x\right\} = \sigma q_{-X_{0,1}}(x) - \mu,$$

and hence

$$\begin{aligned}\rho_{\Psi_t}(X_{\mu,\sigma}) &= \int_0^1 q_{-X_{\mu,\sigma}}(u)\,\Psi_t'(1-u)\,du = \int_0^1 \left(\sigma q_{X_{0,1}}(u) - \mu\right)\Psi_t'(1-u)\,du \\ &= \sigma\rho_t(X_{0,1}) - \mu\left(\Psi_t^{(i)}(1) - \Psi_t(0)\right) = \sigma\rho_t(X_{0,1}) - \mu,\end{aligned}$$

because, by construction, $\Psi_t(1) = 1$ and $\Psi_t(0) = 0$. So we just need to focus on the centered Gaussian distribution. It is convenient to write it in the following form:

$$\forall \mu, \tilde{\mu}, \sigma, \tilde{\sigma},\ \forall t \geq 0,\ \rho_{\Psi_t}(X_{\tilde{\mu},\tilde{\sigma}}) = \tilde{\sigma}\rho_{\Psi_t}(X_{\mu,\sigma})/\sigma + \mu\tilde{\sigma}/\sigma - \tilde{\mu}.$$

3.2.1. *AIMIN distortion for Gaussian distribution.*

Lemma 3.2. *Let $X \hookrightarrow \mathcal{N}(0,1)$ and Ψ_t be an AIMIN concave distortion semigroup. Then $t \mapsto \rho_{\Psi_t}(X)$ is an increasing function.*

Proof. The proof is similar to the proof above. For ease of notation, let $\theta = \exp(t)$:

$$\begin{aligned}\forall \theta \geq 1,\ \partial_\theta \rho_{\Psi_\theta^{(2)}}(X) &= \int_0^1 q_{-X}(u)\,u^{\theta-1}\left[1+\log(u)\right]du = \int_0^{1/2} q_{-X}(u)\,u^{\theta-1}\left[1+\log(u)\right]du \\ &- \int_0^{1/2} q_{-X}(v)(1-v)^{\theta-1}\left[1+\log(1-v)\right]dv = \int_0^{1/2} q_{-X}(u)\left[g_\theta(u) - g_\theta(1-u)\right]du,\end{aligned}$$

where $g_\theta : u \mapsto u^\theta(1+\log(u))$. We have, $\forall\theta \geq 1,\ \forall u \in [0,1/2), g_\theta(u) - g_\theta(1-u) < 0$, and the lemma follows. □

Remark 3.3. Let $X \hookrightarrow \mathcal{N}(\mu,\sigma)$ and Ψ_t as above. If $\mu > 0$ then $\alpha(X) \in \mathbb{R}_+^*$, otherwise $\alpha(X) = \infty$.

We consider a centered Gaussian distribution $\mathcal{N}\left(0,\sigma^2\right)$ and we have (we expand the erf^{-1} function up to order $n = 2p+1$:

$$\begin{aligned}\forall t \geq 0,\ \rho_{\Psi_t^{(2)}}(X_{\sigma^2}) &= \sigma\sqrt{2}\int_0^1 \text{erf}^{-1}(2u-1)\,\Psi_t'^{(i)}(1-u)\,du = \sigma e^t\sqrt{2}\int_0^1 \text{erf}^{-1}(2u-1)\,u^{e^t-1}du \\ &= \sigma e^t\sqrt{2}\int_0^1\left(\sum_{i=0}^p \beta_{2i+1}(2u-1)^{2i+1} + \mathcal{O}\left(u^{2p+1}\right)\right)u^{e^t-1}du \\ &\approx -\sigma e^t\sqrt{2}\sum_{i=0}^p \beta_{2i+1}\sum_{k=0}^{2i+1}\frac{C_{2i+1}^k(-2)^k}{k+e^t}.\end{aligned}$$

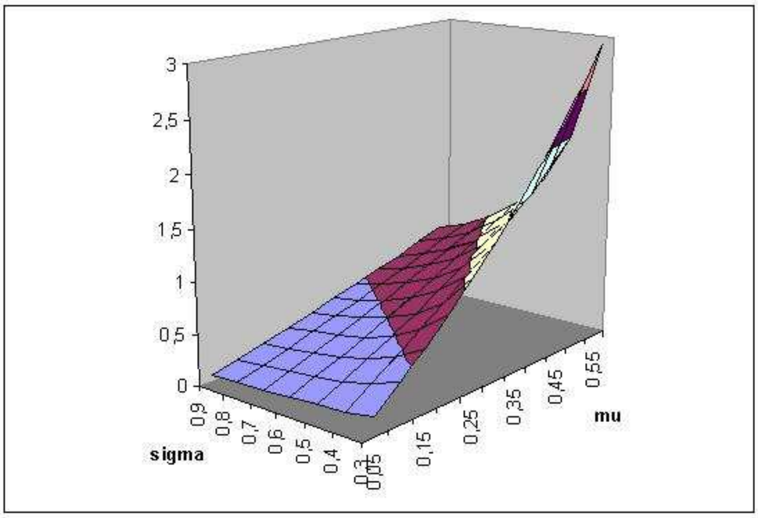

FIGURE 1. Acceptability index $\alpha(X_{\mu,\sigma})$ for the AIMAX distortion

3.2.2. *Numerical Implementation and issues.* We have proved that, for any $\mu > 0, \sigma > 0$, $\alpha(X_{\mu,\sigma})$ exists and is unique. The graphs below present some numerical results for this acceptability index. Some comments arise from the numerical implementation:

- The implementation has been carried out in Python.
- The number of steps needed for a stable integration heavily depends on the region defined by the couple (μ, σ). We show hereafter some numerical caveats.

From Figure 2, we observe that, for some range of (μ, σ), we do not need too many steps for the integration, as we are only interested by the point $t^* : \rho_{\Psi_{t^*}}(X) = 0$. A larger value of μ means a downward translation of the graphs. So, for large values of μ, the integration requires a very large number of steps, and hence the minimisation procedure will be much longer.

Remark 3.4. We provide an intuitive reason for the shape of these graphs, in particular when t gets large. The graphs correspond to the function $t \mapsto \int_{[0,1]} q_{-X}(u)\,\Psi_t'(1-u)\,du$. $\Psi_t'(.)$ is a strictly decreasing function from $+\infty$ to 0 (Ψ_t is strictly concave). The higher the t, the more concave the function Ψ_t, and hence, in the integral, the higher the weight in the upper side of the integral. Hence, if the number of steps in the integration is not large enough, we omit many terms, in particular many large terms. For more accurate results, we should adopt an adaptive numerical scheme, specifying a number of steps proportional to the concavity of Ψ_t at each point. However, for our purposes, and in particular, with the values we chose for the parameters, this is not of fundamental importance, as we are just interested by the point where the function vanishes. But for some values of the parameters, this integration might be computationally really intensive.

3.3. **Some analytical results for the Student distribution.** In some cases, the inverse cdf of the Student is available in closed-form, see [17] for the details. In particular,

- for $n = 1$, this is the Cauchy distribution and $F^{\leftarrow}(u) = tan(\pi(u - 1/2))$.
- for $n = 2$, $F^{\leftarrow}(u) = \frac{2u-1}{\sqrt{2u(1-u)}}$. This case is less interesting as it has infinite variance.

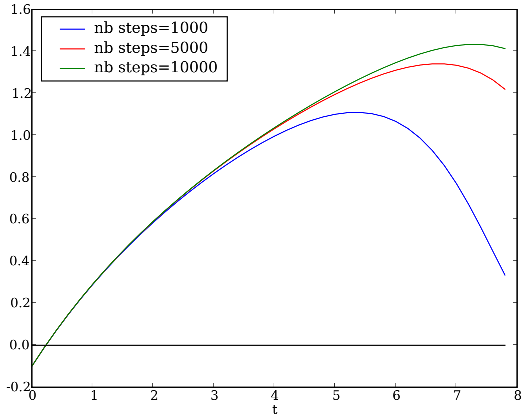

FIGURE 2. AIMIN with $\mathcal{N}(\mu = 0.1, \sigma = 0.5)$

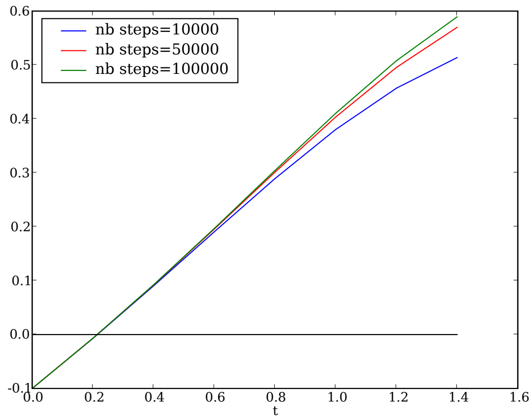

FIGURE 3. AIMAX with $\mathcal{N}(\mu = 0.1, \sigma = 0.5)$

- for $n = 4$, $F^{\leftarrow}(u) = \operatorname{sgn}(u - 1/2)\sqrt{\frac{2}{\sqrt{u(1-u)}}\cos\left(\frac{1}{3}\arccos\left(2\sqrt{u(1-u)}\right)\right) - 4}$. This case is particularly interesting as it has finite variance and infinite kurtosis.

We note that, as Student distributions are symmetric around 0, we have, for all ν, $q_{X_\nu} = q_{-X_\nu}$. We study here the case $\nu = 2$ (infinite variance). For a concave distortion semigroup $(\Psi_t(.))_{t\geq 0}$, we then have

$$\forall t \geq 0,\ \rho_{\Psi_t}(X_2) = \int_0^1 q_{X_2}(u)\, \Psi_t'(1-u)\, du.$$

Let us consider first the AIMIN distortion:

$$\forall t \geq 0,\ \rho_{\Psi_t^{(2)}}(X_2) = \mathrm{e}^t \int_0^1 \frac{2u-1}{\sqrt{2u(1-u)}} u^{\exp(t)-1} du = \frac{\sqrt{2\pi}\Gamma(1/2 + \mathrm{e}^t)}{\Gamma(1+\mathrm{e}^t)} - \frac{\Gamma(\mathrm{e}^t - 1/2)}{\Gamma(\mathrm{e}^t)}\sqrt{\pi/2}.$$

Figure 4 plots $\rho_{\Psi_t^{(2)}}$ as a function of t. We observe that, contrary to the Normal distribution, it is not strictly increasing. As we proved above, $\forall t > 0,\ \rho_{\Psi_t^{(2)}}(X_2) > 0$, we also have $\alpha(X_2) = \infty$.

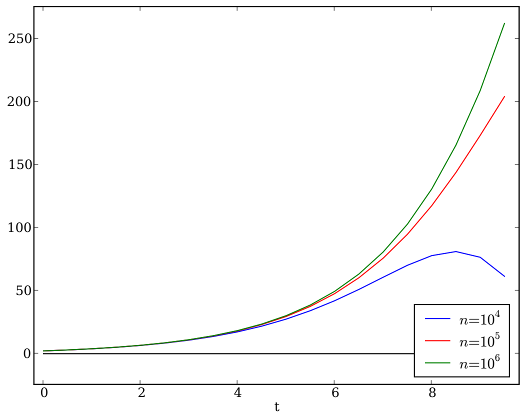

FIGURE 4. AIMAX risk measure for a Student distribution with $\nu = 2$, and n number of steps for the integration

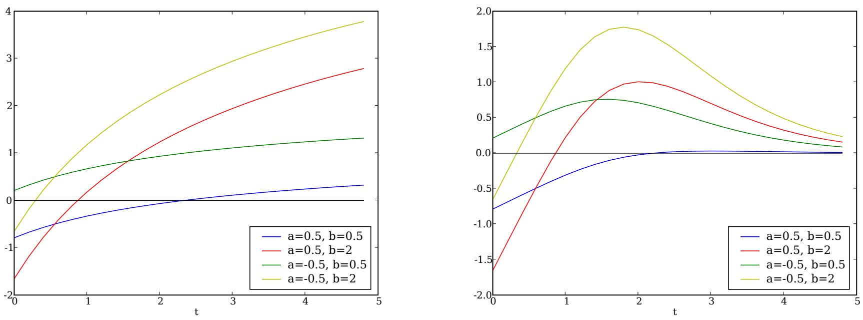

FIGURE 5. AIMIN (left) and AIMAX (right) for the Gumbel distribution with 10^3 steps.

3.4. **An asymmetric distribution: Gumbel.** We now consider an asymmetric distribution. The reason for choosing the Gumbel distribution is that the inverse cdf is known is closed form. Indeed, let $X \hookrightarrow \mathcal{G}(a, b)$, with $a \in \mathbb{R}$, $b > 0$, we have

$$\forall u \in (0, 1), \ q_X(u) = a - b \log(-\log(u)), \quad \text{and } q_{-X}(u) = -a + b \log(-\log(1-u)).$$

We now consider the AIMIN spectral risk measure

$$\forall t \geq 0, \ \rho_{\Psi_t}(X) = \mathrm{e}^t \int_0^1 [-a + b \log(-\log(1-u))] u^{\mathrm{e}^t - 1} du = -a + b\mathrm{e}^t \int_0^1 \log(-\log(1-u)) u^{\mathrm{e}^t - 1} du$$

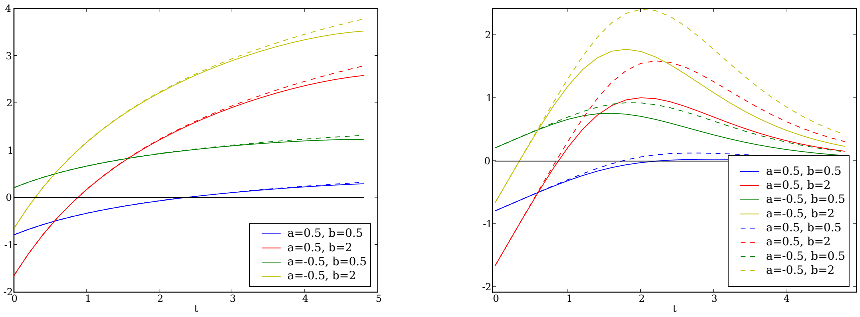

FIGURE 6. Left: Numerical issues for AIMIN with the Gumbel distribution; solid line: 10^3 steps, dashed line: 10^4 steps. Right: Numerical issues for AIMAX with the Gumbel distribution; solid line: 10^4 steps, dashed line: 10^6 steps

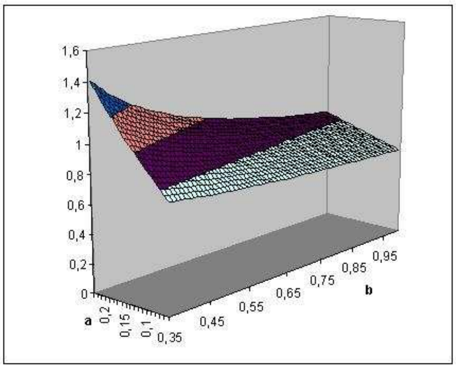

FIGURE 7. Optimal $\alpha(X)$ when $X \hookrightarrow \mathcal{G}(a,b)$, with 10^4 steps

APPENDIX

APPENDIX A. OVERVIEW OF THE DISTRIBUTIONS USED

A.1. **Student.** $X \hookrightarrow \mathcal{S}(\nu)$ for $\nu \in \mathbb{N}$.

$$\forall x \in \mathbb{R},\ f(x) = \frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu\right)}{\sqrt{\pi\nu}\,\Gamma\left(\frac{1}{2}\nu\right)\left(1+\frac{x^2}{\nu}\right)^{\frac{1}{2}+\frac{1}{2}\nu}},$$

$$F(x) = \begin{cases} \frac{1}{2} + \frac{\arctan\left(\frac{x}{\sqrt{\nu}}\right)}{\pi} + \frac{x\sqrt{\nu}}{\pi(\nu+x^2)}\left[\frac{\arcsin\left(\sqrt{\frac{\nu}{\nu+x^2}}\right)}{\sqrt{\frac{\nu}{\nu+x^2}}\sqrt{1-\frac{\nu}{\nu+x^2}}} - \frac{1}{2}\frac{\Gamma\left(\frac{1}{2}+\frac{\nu}{2}\right)\sqrt{\pi}H\left(\left[1,\frac{1}{2}+\frac{\nu}{2}\right],\left[1+\frac{\nu}{2}\right],\frac{\nu}{\nu+x^2}\right)}{\Gamma\left(\frac{1}{2}+\frac{\nu}{2}\right)\left(1+\frac{x^2}{\nu}\right)^{\nu/2-1/2}}\right], & \text{if } \nu = 1 \\ \frac{1}{2} + \frac{x}{2\sqrt{\nu+x^2}}\left[\sqrt{\frac{\nu+x^2}{x^2}} - \frac{\Gamma\left(\frac{1}{2}+\frac{\nu}{2}\right)H\left(\left[1,\frac{1}{2}+\frac{\nu}{2}\right],\left[1+\frac{\nu}{2}\right],\frac{\nu}{\nu+x^2}\right)}{\sqrt{\pi}\Gamma\left(\frac{1}{2}+\frac{\nu}{2}\right)\left(1+\frac{x^2}{\nu}\right)^{\nu/2}}\right], & \text{otherwise} \end{cases}$$

where $H([u],[l],x)$ represents the Generalised Hypergeometric function with upper parameters u and lower parameters l, evaluated at x.

A.2. **Gumbel.** $X \hookrightarrow \mathcal{G}(a,b)$ for $a \in \mathbb{R},\ b > 0$. a is a location parameter, and b a scale parameter. We have

$$\forall x \in \mathbb{R},\ f(x) = \frac{1}{b}\mathrm{e}^{-\frac{x-a}{b}}\mathrm{e}^{-\mathrm{e}^{-\frac{x-a}{b}}},\ F(x) = \mathrm{e}^{-\mathrm{e}^{-\frac{x-a}{b}}},\quad \text{and } \mathbb{E}(X) = a + \gamma b,\ \mathbb{E}(X) = \frac{1}{6}b^2\pi^2,$$

where γ is the Euler constant.

Appendix B. Families of distortion functions

In the following, we consider the following distortion functions (for $x \in [0,1]$, the families are indexed by t):

- $\psi_t^{(1)}(x) = x\mathrm{e}^t \wedge t$: AIW index
- $\psi_t^{(2)}(x) = 1 - (1-x)^{\mathrm{e}^t}$: AIMIN index
- $\psi_t^{(3)}(x) = x\mathrm{e}^{-t}$: AIMAX index
- $\psi_t^{(4)}(x) = \frac{\mathrm{e}^t}{1-\mathrm{e}^t}\left(\mathrm{e}^{-tx} - 1\right)$: Exponential utility function

Appendix C. Taylor series expansion for the erf^{-1} function

We know that

$$\mathrm{erf}(x) = \frac{1}{2}\sqrt{\pi}x + \frac{1}{24}\pi^{\frac{3}{2}}x^3 + \frac{7}{960}\pi^{\frac{5}{2}}x^5 + \frac{127}{80640}\pi^{\frac{7}{2}}x^7 + \frac{4369}{11612160}\pi^{\frac{9}{2}}x^9 + \mathcal{O}\left(x^{11}\right),\quad \text{for } n = 10.$$

For higher orders, one can use the algorithm by Steinbrecher in [18]:

$$\mathrm{erf}^{-1}(x) = \sum_{n\geq 0}\frac{\gamma_n}{2n+1}\left(\frac{\sqrt{\pi}}{2}x\right)^{2n+1},\quad \text{where } \gamma_0 = 1 \text{ and } \forall n \geq 1,\ \gamma_n = \sum_{k=0}^{n-1}\frac{\gamma_k\gamma_{n-k-1}}{(k+1)(2k+1)}.$$

References

[1] P. Artzner, F. Delbaen, F. Eber, J.M. Heath. Coherent measures of risk. *Mathematical Finance* 9, No. 3, 203-228. (1999)

[2] A. Cherny, D. Madan. On measuring the degree of market efficiency. *Preprint*(2007)

[3] A. Cherny, D. Madan. Pricing and hedging in incomplete markets with coherent risk. *Preprint*(2006)

[4] A. Cherny, D. Madan. CAPM, Rewards, and empirical asset pricing with coherent risk. *Preprint*(2006)

[5] A. Cherny. Weighted VaR and its properties. *Finance and Stochastics*, 10 (2006), No. 3, p. 367-393

[6] A. Cherny. Pricing with coherent risk. *Probability Theory and Its Applications*, 52 (2007), No. 3, p. 506-540.

[7] A. Cherny, D. Madan. Recovering coherent risk aversion and scaling consistency. *Preprint*(2008)

[8] A. Cherny, D. Madan. New Measures for Performance Evaluation. *Preprint*(April 2008)

[9] A. Cherny, D. Filipovic. Concave Distortion Semigroups. *Preprint*(2007)

[10] F. Delbaen. Coherent Risk Measures on general probability spaces. *Advances in Finance and Stochastics.* Springer 2002, 1-37.

[11] N. Dunford, J. Schwartz. Linear Operators. Part I: General Theory. *New York: Interscience Publishers*, 1958.

[12] E. Eberlein, R. Frey, E.A. von Hammerstein. Advanced credit portfolio modeling and CDO pricing. *Preprint, University of Freiburg*, 2007.

[13] D. Filipovic, G. Svindland, Convex Risk Measures on L^p spaces. *Preprint* , July 2007.

[14] H. Follmer, A. Schied, Stochastic Finance: An Introduction in discrete Time, 2nd Edition. *De Gruyter Studies in Mathematics* (2004).

[15] R.B. Israel, J.S. Rosenthal, J.Z. Wei. Finding generators for Markov chains via empirical transition matrices, with applications to credit ratings. *Mathematical Finance*, 11 (2001), p. 245-265.

[16] M. Kaina, L. Ruschendrof, On convex risk measures on L^p-spaces. To appear in *Mathematical Methods in Operations Research*, 2007.

[17] W. Shaw. New Methods for Managing "Student's" T Distribution. *Preprint King's College* (2006).

[18] W. Shaw, G. Steinbrecher. Quantile Mechanics. *Preprint King's College*(2008).

Department of Mathematics, Imperial College London and Zeliade Systems, Paris

E-mail address: antoine.jacquier@alumni.essec.fr