Financial Models with Defaultable Numéraires

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Joint work with Travis Fisher and Sergio Pulido

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Lack of a natural numéraire

- Standard models of financial markets: in units of a pre-specified numéraire.
- Here: multiple financial assets, any of which may potentially lose all value relative to the others.

Contribution

1. (Formulation of the First and Second FTAP. Symmetric in the sense that no asset is prioritized.)
2. Interpretation of strict local martingale models, arising by fixing a numéraire that has positive probability to default.
   $\Rightarrow$ Non-classical pricing formulas can be economically justified and extended.
3. Assume that for each asset there exists a probability measure under which discounted prices (with the corresponding asset as numéraire) are local martingales. These measures need not be equivalent.
   Question: How can these measures be *aggregated* to an arbitrage-free pricing operator that takes all events of devaluations into account?

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Non-classical pricing operators

- Popular model in FX:

$$S_{1,2}(t) = S_{1,2}(0) + \int_0^t \left(aS_{1,2}(u)^2 + bS_{1,2}(u) + c\right) \mathrm{d}W(u)$$

  "Quadratic normal volatility" (stopped when hitting zero)
- Calibration usually yields strict local martingale dynamics.
- Let's assume a complete market and zero interest rate.
- Superreplication cost of $S_{1,2}(T)$ is strictly smaller than $S_{1,2}(0)$ (if we price according to risk-neutral expectations). This contradicts no-arbitrage "in practice."
- Possible ways out:
  - Use a different model.
  - Change the concept of pricing operator.

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New pricing operators

- Lewis: "add correction term" to risk-neutral expectation when pricing calls.
- Madan & Yor: Exchange expectations and limits.
- Cox & Hobson: Restrict class of admissible strategies.
- Paulot: Linear operator on a Banach space of payoffs
- Carr & Fisher & Ruf:
  - Note that a change of numéraire via strict local martingale $S_{1,2}$ yields non-equivalent measure.
  - Then consider the minimal superreplication cost under both measures (the original one and the new one).
  - Yields an explicit formula for the correction term.

Issues:

- Correction term seems non-symmetric in currencies.
- What to do in an incomplete market??
- What to do with more than two currencies??

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Relative prices are modelled by an exchange matrix

- d: number of currencies
- Let $S_{i,j}(t)$ denote the price of the j:th currency in terms of the i:th currency, at time t.
- $S = (S_{i,j})$ is an $\mathbb{F}$–progressive, càdlàg process taking values in $[0, \infty]^{d \times d}$ such that $S(t)$ is an exchange matrix:

$$S_{i,j}(t)S_{j,k}(t) = S_{i,k}(t) \quad \text{(whenever defined);}$$
$$S_{i,i}(t) = 1.$$

- Note: there exists always a *strongest* currency i^* with $\sum_j S_{i*,j}(t) \leq d$.
- Define: $\mathfrak{A}(t) = \{i : \sum_j S_{i,j}(t) < \infty\} \neq \emptyset$.

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Value vector

- A value vector $v = (v_i)_i$ (with respect to $S(t)$) encodes the price of an asset in terms of the d currencies.
- The i:th component describes the price of an asset in terms of the i:th currency.
- v satisfies consistency condition:

$$S_{i,j}(t)v_j = v_i \quad \text{(whenever defined).}$$

- $\mathcal{D}^t$: the set of all $\mathcal{F}(t)$–measurable value vectors with respect to $S(t)$ (which are bounded, in a weak sense)

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Valuation operator

- A *valuation operator* relates future random prices to present deterministic prices.
- Concept goes back to Harrison & Pliska (1981); see also Biagini & Cont (2006) and literature on risk measures.

We say that a family of operators $\mathbb{V} = (\mathbb{V}^{r,t})_{0 \leq r \leq t \leq T}$, with

$$\mathbb{V}^{r,t} : \mathcal{D}^t \to \mathcal{D}^r,$$

is a valuation operator with respect to S if it satisfies:

1. Positivity
2. Linearity
3. Continuity from below
4. Time consistency
5. Martingale property
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Valuation operator — the conditions

1. (Positivity) If $C \in \mathcal{D}^T$ and $C \geq 0$ then $\mathbb{V}^{0,T}(C) \geq 0$.
2. (Linearity) If $H \in \mathcal{L}^{\infty,r}$, and $C, C' \in \mathcal{D}^t$ then

$$\mathbb{V}^{r,t}(H\mathbf{1}_{\{H\neq 0\}}C + C') = H\mathbf{1}_{\{H\neq 0\}}\mathbb{V}^{r,t}(C) + \mathbb{V}^{r,t}(C').$$

3. (Continuity from below) If $(C_n)_{n\in\mathbb{N}} \subset \mathcal{D}^T$ is a nondecreasing sequence of nonnegative value vectors converging to $C \in \mathcal{D}^T$, then $\mathbb{V}^{0,t}(C_n)$ converges to $\mathbb{V}^{0,t}(C)$.
4. (Time consistency) For $C \in \mathcal{D}^T$,

$$\mathbb{V}^{r,t}(\mathbb{V}^{t,T}(C)) = \mathbb{V}^{r,T}(C).$$

5. (Martingale property) $\mathbb{V}^{t,T}(S_{\cdot,i}(T)) = S_{\cdot,i}(t)\mathbf{1}_{\{i\in\mathfrak{A}(t)\}}$.
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Disaggregation and aggregation

A family $(\mathbb{Q}_i)_i$ of probability measures such that S_i a $\mathbb{Q}_i$–supermartingale is called *consistent* if the following change-of-numéraire formula holds:

$$\mathbb{E}^{\mathbb{Q}_i}[S_{i,j}(t)\mathbf{1}_A] = S_{i,j}(0) \times \mathbb{Q}_j(A \cap \{S_{j,i}(t) > 0\}).$$

Given a valuation operator $\mathbb{V}$ there exist a consistent family of supermartingale measures $(\mathbb{Q}_i)_i$ such that

$$\mathbb{V}_j^{r,t}(C) = \sum_i S_{j,i}(r)\mathbb{E}_r^{\mathbb{Q}_i}\left[\frac{C_i}{|\mathfrak{A}(t)|}\right] \tag{1}$$

for all $r \leq t$, $j \in \mathfrak{A}(r)$, $C \in \mathcal{D}^t$.
Conversely, given a consistent family of supermartingale measures $(\mathbb{Q}_i)_i$, (1) defines a valuation operator $\mathbb{V}$.

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The appearance of strict local martingales

Consistent family $(\mathbb{Q}_i)_i$, with $A = \Omega$:

$$\mathbb{E}^{\mathbb{Q}_i}[S_{i,j}(t)] = S_{i,j}(0) \times \mathbb{Q}_j(S_{j,i}(t) > 0).$$

- $S_{i,j}$ is a $\mathbb{Q}_i$–martingale if and only if $\mathbb{Q}_j(S_{j,i}(T) = 0) = 0$.
- $S_{i,j}$ is a $\mathbb{Q}_i$–local martingale if and only if $S_{j,i}$ does not jump to zero under $\mathbb{Q}_j$.

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The case of two assets

$d = 2$, with value vector $C = (C_1, C_2)^{\mathrm{T}}$

E.g., $C = ((S_{1,2}(T) - K)^+, (1 - KS_{2,1}(T))^+)^{\mathrm{T}}$

$$\mathbb{V}_j^{0,T}(C) = S_{j,1}(0) \times \mathbb{E}^{\mathbb{Q}_1}\left[\frac{C_1}{|\mathfrak{A}(T)|}\right] + S_{j,2}(0) \times \mathbb{E}^{\mathbb{Q}_2}\left[\frac{C_2}{|\mathfrak{A}(T)|}\right]$$
$$= S_{j,1}(0) \times \mathbb{E}^{\mathbb{Q}_1}[C_1] + \underbrace{S_{j,2}(0) \times \mathbb{E}^{\mathbb{Q}_2}[C_2 \mathbf{1}_{\{S_{1,2}(T)=\infty\}}]}_{\text{equals Lewis' correction term}}$$

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$$\begin{aligned}\mathbb{V}_j^{0,T}(C) &= S_{j,1}(0) \times \mathbb{E}^{\mathbb{Q}_1}\left[\frac{C_1}{|\mathfrak{A}(T)|}\right] + S_{j,2}(0) \times \mathbb{E}^{\mathbb{Q}_2}\left[\frac{C_2}{|\mathfrak{A}(T)|}\right] \\ &= S_{j,1}(0) \times \mathbb{E}^{\mathbb{Q}_1}[C_1] + \underbrace{S_{j,2}(0) \times \mathbb{E}^{\mathbb{Q}_2}[C_2 \mathbf{1}_{\{S_{1,2}(T)=\infty\}}]}_{\text{equals Lewis' correction term}}\end{aligned}$$

Example: The Camara-Heston model

- Câmara-Heston extend the BSM model with a huge jump upward or a huge jump downward to explain observed skews and smiles.
- They derive analytic call and put prices by solving a suitable PDE.
- In our setup: $d = 2$
- W is $\mathbb{P}$–BM, and τ_1, τ_2 are independent exponential times with intensities λ_1, λ_2:

$$S_{1,2}(t) = \mathrm{e}^{\sigma W(t)+\mu t}\mathbf{1}_{\{t\le\tau_1\wedge\tau_2\}} + \infty \times \mathbf{1}_{\{\tau_1<\tau_2\wedge t\}}$$

- Call option with $C_1 = (S_{1,2}(T) - K)^+$ and $C_2 = (1 - KS_{2,1}(T))^+$. Then

$$\mathbb{V}_1^{0,T}(C) = \mathrm{e}^{-\lambda_1 T}S_{1,2}(0)\Phi(d_1) - K\mathrm{e}^{-\lambda_2 T}\Phi(d_2) + S_{1,2}(0)(1-\mathrm{e}^{-\lambda_1 T}).$$

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The concept of “no obvious devaluations”

We say that a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F}(T))$ satisfies “No Obvious Devaluations” (NOD) if

$$\mathbb{P}(i \in \mathfrak{A}(T) | \mathcal{F}(\tau)) > 0 \text{ on } \{\tau < \infty\} \cap \{i \in \mathfrak{A}(\tau)\}$$

for all i and stopping times τ.

Aggregation without numéraire-consistency

Let $(\mathbb{Q}_i)_i$ be be a family of probability measures. Then there exists a martingale valuation operator $\mathbb{V} \sim \sum_i \mathbb{Q}_i$ if one of the following two conditions is satisfied:

1. S_i is a $\mathbb{Q}_i$–martingale.
2. The following four conditions hold:
   2.1 S_i is a $\mathbb{Q}_i$–local martingale.
   2.2 $\sum_i \mathbb{Q}_i$ satisfies (NOD).
   2.3
   $$\mathbb{Q}_k\big|_{\mathcal{F}\cap\{\sum_j S_{k,j}(T)<\infty\}} \sim \left(\sum_i \mathbb{Q}_i\right)\Bigg|_{\mathcal{F}\cap\{\sum_j S_{k,j}(T)<\infty\}}.$$
   2.4 There exist $\epsilon > 0$, $N \in \mathbb{N}$, predictable times $(T_n)_{n\in\{1,\cdots,N\}}$ s.t.
   $$\bigcup_k \left\{(t,\omega) : \sum_j S_{k,j}(t) = \infty \text{ and } \sum_j S_{k,j}(t-) \leq d + \varepsilon\right\} \subset \bigcup_{n=1}^{N} [\![T_n]\!].$$

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Conclusion

- We consider an exchange economy with d currencies, where each currency has the possibility to complete devaluate against any other currency.
- (In such an economy, we introduce the concept of a valuation operator and link it to a no-arbitrage condition.)
- We interpret the lack of martingale property of an asset price as a reflection of the possibility that the numéraire currency may devalue completely.
- We study conditions under which not necessarily equivalent measures, corresponding to different numéraires, may be aggregated to obtain a numéraire-independent valuation operator.

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Merci beaucoup!

Many thanks!

Bon Appétit!