Generalized Dynkin games with g-conditional expectation and nonlinear pricing of game options

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References

- *Classical Dynkin Games in continuous time*: Alario-Nazaret, Lepeltier and Marchal, B. (1982).
- Links with Doubly RBSDEs when the driver g **does not depend on** y, z: Cvitanic and Karatzas (1996), Hamadène (2002) and Lepeltier (2000) (Hyp: Brownian+ regularity).
- Pricing of Game options, links with Dynkin Games : Kifer (2000)
- Pricing of Game options in a complete financial market and links with Doubly RBSDEs with a driver g **linear with respect to** y, z: Hamadène (2006).
- Doubly RBSDEs with jumps: e.g. Essaky, Harraj, Ouknine (2005), Hamadène and Hassani (2006), Crépey and Matoussi (2008).
- This work : *Generalized Dynkin games and DRBSDEs* *http://arxiv.org/abs/1504.06094* 2013
  \+ *Nonlinear pricing in a market with defaults*: forthcoming.

Framework

Let $(\Omega, \mathcal{F}, P)$ be a probability space.

- Let W be a Brownian motion
- $N(dt, du)$ be a Poisson random measure with intensity $\nu(du)dt$ such that ν is a σ-finite measure on $\mathbf{R}^*$.
  Let $\tilde{N}(dt, du)$ be its compensated process.
- Let $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ be the natural filtration associated with W and N.
- Fix $T > 0$.

Notation

- $\boldsymbol{H}^2$: set of predictable processes ϕ s.t. $\|\phi\|^2_{\boldsymbol{H}^2} := E\left[\int_0^T \phi_t^2 dt\right] < \infty$.
- L^2_ν: set of Borelian fns ℓ s.t. $\|\ell\|^2_\nu := \int_{\mathbf{R}^*} |\ell(u)|^2 \nu(du) < +\infty$.
  L^2_ν is a Hilbert with $\langle \delta, \ell \rangle_\nu := \int_{\mathbf{R}^*} \delta(u)\ell(u)\nu(du)$
- $\boldsymbol{H}^2_\nu$: set of predictable processes l s.t.
  $\|l\|^2_{\boldsymbol{H}^2_\nu} := E\left[\int_0^T \|l_t\|^2_\nu dt\right] < \infty$.
- S^2: set of real-valued RCLL adapted processes ϕ s.t.
  $\|\phi\|^2_{S^p} := E(\sup_{0\leq t\leq T} |\phi_t|^2) < \infty$.
- $\mathcal{T}_0$: set of stopping times τ s.t. $\tau \in [0, T]$ a.s
- For S in $\mathcal{T}_0$, $\mathcal{T}_S := \{\tau\,,\, S \leq \tau \leq T \text{ a.s.}\}$

BSDEs with jumps

Definition: A function g is a *driver* if $g : \Omega \times [0,T] \times \mathbf{R}^2 \times L^2_\nu \to \mathbf{R}$ $(\omega,t,y,z,k) \mapsto g(\omega,t,t,y,z,k)$ is predictable, and $g(.,0,0,0) \in \boldsymbol{H}^2$.

A driver g is a *Lipschitz driver* if $\exists\ C \geq 0$ s.t.

$$|g(\omega,t,y_1,z_1,k_1)-g(\omega,t,y_2,z_2,k_2)| \leq C(|y_1-y_2|+|z_1-z_2|+\|k_1-k_2\|_\nu).$$

$\forall\ (y_1,z_1,k_1),\ \forall\ (y_2,z_2,k_2)$

Theorem

(Barles-Buckdahn-Pardoux) Let $T>0$. Let $\xi \in \mathcal{L}^2(\mathcal{F}_T)$,
$\exists\ !\ (X,Z,k) \in S^{2,T} \times \boldsymbol{H}^{2,T} \times \boldsymbol{H}^{2,T}_\nu$ s.t.

$$-dX_t = g(t,X_{t^-},Z_t,k_t)dt - Z_t dW_t - \int_{\mathbf{R}^*} k_t(e)\tilde{N}(dt,de); \quad Y_T = \xi.$$

This solution is denoted by $(X^g(\xi,T), Z^g(\xi,T), k^g(\xi,T))$.

Nonlinear pricing associated with g/ g-evaluation

3 assets: prices S^0, S^1, S^2 with $dS^0_t = S^0_t r_t dt$

$$\begin{cases} dS^1_t = S^1_t[\mu^1_t dt + \sigma^1_t dW_t] \\ dS^2_t = S^{;2}_t[\mu^2_t dt + \sigma^2_t dW_t + \beta_t d\tilde{N}_t]. \end{cases}$$

Let $x =$ initial wealth.
At t, he chooses the amount φ^1_t (resp. φ^2_t) invested S^1 (resp S^2).
$\varphi_. = (\varphi^1_t, \varphi^2_t)'$ called *risky assets stategy*.
Let $V^{x,\varphi}_t$ (or V_t) $=$ value of the portfolio.
In the classical case

$$dV_t = \quad (r_t V_t + \varphi^1_t \theta^1_t \sigma^1_t + \varphi^2_t \theta^2_t \beta_t)dt + \varphi'_t \sigma_t dW_t + \varphi^2_t \beta_t d\tilde{N}_t,$$

where $\theta^1_t := \dfrac{\mu^1_t - r_t}{\sigma^1_t}$ and $\theta^2_t := \dfrac{\mu^2_t - \sigma^2_t \theta^1_t - r_t}{\beta_t}$.

Case with nonlinear constraints:

$$-dV_t = g(t, V_t, {\varphi_t}'\sigma_t, \varphi_t^2\beta_t)dt - {\varphi_t}'\sigma_t dW_t - \varphi_t^2\beta_t d\tilde{N}_t,$$

or equivalently, setting $Z_t = {\varphi_t}'\sigma_t \quad K_t = \varphi_t^2\beta_t,$

$$-dV_t = g(t, V_t, Z_t, K_t)dt - Z_t dW_t - K_t d\tilde{N}_t,$$

Consider a European option with payoff $\xi \in L^2(\mathcal{F}_T)$.
$\exists\, !\ (X, Z, K)$ square integrable/

$$-\,dX_t = g(t, X_t, Z_t, K_t)dt - Z_t dW_t - K_t d\tilde{N}_t;\ X_T = \xi. \qquad (1)$$

The hedging risky assets stategy $\varphi = (\varphi^1, \varphi^2)'$ is such that

$${\varphi_t}'\sigma_t = Z_t \ ; \ \varphi_t^2\beta_t = K_t, \qquad (2)$$

$\Rightarrow X = V^{X_0,\varphi}$ (value of the replicating portfolio) = price.
Example:
$g(t, V_t, \varphi_t\sigma_t, \varphi_t^2\beta_t) = -(r_t V_t + \varphi_t^1\theta^1\sigma^1 + \varphi_t^2\theta^2\beta) + \rho(\varphi_t^1 + \varphi_t^2)^+.$

- This defines a *nonlinear pricing* system, introduced in El Karoui-Q (1996) in a Brownian framework, called g-*evaluation* by Peng (2004), denoted by $\mathcal{E}^g$.
- $\forall\ T$, $\forall\ \xi \in L^2(\mathcal{F}_T)$, the g-*evaluation* of (T, ξ) is defined by

$$\mathcal{E}^g_{t,T}(\xi) := X^g_t(T, \xi),\ 0 \leq t \leq T.$$

- Definition

  An RCLL adapted process X_t in $\mathcal{S}^2$ is said to be an $\mathcal{E}^g$-**supermartingale** if $\mathcal{E}_{\sigma,\tau}(X_\tau) \leq X_\sigma$ a.s. , $\forall\ \sigma \leq \tau \in \mathcal{T}_0$.

  - Note that $\forall\ x \in \mathbb{R}\ \forall\ \varphi$, $V^{x,\varphi}$ is an $\mathcal{E}^g$-martingale ("g-martingale").
  - In order to ensure that $\xi \mapsto \mathcal{E}^g_{\cdot,T}(\xi)$ is non decreasing, we make the following assumption:

$$g(t, y, z, k_1) - g(t, y, z, k_2) \geq \gamma_t^{y,z,k_1,k_2}\ (k_1 - k_2)\nu_t,$$

$$\gamma_t^{y,z,k_1,k_2} \geq -1.$$

Assumption A.1

$\forall\ (y, z, k_1, k_2)$,

$$g(t, y, z, k_1) - g(t, y, z, k_2) \geq \langle \gamma_t^{y,z,k_1,k_2}\ ,\ k_1 - k_2 \rangle_\nu,$$

with $\quad \gamma : [0, T] \times \Omega \times \mathbf{R}^2 \times (L^2_\nu)^2 \to L^2_\nu\ ;\ (\omega, t, y, z, k_1, k_2) \mapsto \gamma_t^{y,z,k_1,k_2}(\omega, .)$
predictable and s.t. $\forall\ (y, z, k_1, k_2)$,

$$\gamma_t^{y,z,k_1,k_2}(e) \geq -1 \quad \text{and} \quad |\gamma_t^{y,z,k_1,k_2}(e)| \leq \psi(e), \qquad (3)$$

where $\psi \in L^2_\nu$.

- $\mathcal{E}^g$ is non decreasing (Q. and Sulem (2013)).

Evaluation of an American option

Let $(\xi_t, 0 \leq t \leq T)$ be a RCLL process $\in \mathcal{S}^2$ (*payoff*)
Price of the **American option**:

$$v(S) := \operatorname{ess\,sup}_{\tau \in \mathcal{T}_S} \mathcal{E}_{S,\tau}(\xi_\tau). \tag{4}$$

Theorem (Sulem, Q. (2013))

(i) We have

$$v(S) = Y_S \quad a.s.$$

where Y is the solution of the **reflected** *BSDE with obstacle ξ.*
(ii) $\tau_\varepsilon := \inf\{u \geq S;\ Y_u \leq \xi_u + \varepsilon\}$ is $K\varepsilon$-optimal for (4), *i.e.*

$$\mathcal{E}_{S,\tau_\varepsilon}(\xi_{\tau_\varepsilon}) \geq Y_S - K\varepsilon \quad \text{a.s.}$$

Result generalized by Grigorova, Quen., Imk.,Ouk. (april 2015) to the case ξ only **right-u.s.c.**

Doob-Meyer Decomposition for $\mathcal{E}$-supermartingales

Theorem : (Dumitrescu-Quenez-Sulem (2014))
(Y_t) be an $\mathcal{E}$-**supermartingale** if and only if $\exists\ (A_t) \in \mathcal{A}^2$ and $(Z, k) \in \boldsymbol{H}^2 \times \boldsymbol{H}^2_\nu$ such that

$$-dY_s = f(s, Y_s, Z_s, k_s)ds + dA_s - Z_s dW_s - \int_{\mathbf{R}^*} k_s(u)\tilde{N}(ds, du).$$

Proof: For each $\tau \in \mathcal{T}_S$, $\quad Y_S \geq \mathcal{E}_{S,\tau}(Y_\tau) \quad a.s.$

$$\Rightarrow \quad Y_S \geq \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_S} \mathcal{E}_{S,\tau}(Y_\tau) \quad a.s.$$

Now, $Y_S \leq \operatorname{ess\,sup}_{\tau \in \mathcal{T}_S} \mathcal{E}_{S,\tau}(Y_\tau)$ a.s.

$$\Rightarrow \quad Y_S = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_S} \mathcal{E}_{S,\tau}(Y_\tau) \quad a.s.$$

By the previous characterization, (Y_t) is equal to the solution of the reflected BSDE with RCLL obstacle (Y_t). $\square$
Generalization by Grigorova, Q. et al. (april 2015):
Mertens Decomposition of strong $\mathcal{E}$-supermartingales (not RCLL)

Evaluation of a Game option

Let ξ and $\zeta \in \mathcal{S}^2$ such that $\xi \leq \zeta$ and $\xi_T = \zeta_T$ a.s.

- The buyer can exercise it at any time $\tau \in \mathcal{T}$. Then, the seller pays to him the amount ξ_τ.
- The seller can cancel it at any $\sigma \in \mathcal{T}$. If $\sigma \leq \tau$, then he pays to the buyer the amount ζ_σ.
- Note that $\zeta_\sigma - \xi_\sigma \geq 0$ is the *penalty* the seller pays for the cancellation of the contract.
- Hence, the game option consists for the seller to select $\sigma \in \mathcal{T}$ and for the buyer to choose $\tau \in \mathcal{T}$, so that the seller pays to the buyer at time $\tau \wedge \sigma$ the payoff

$$I(\tau, \sigma) := \xi_\tau \mathbf{1}_{\tau \leq \sigma} + \zeta_\sigma \mathbf{1}_{\sigma < \tau}. \quad (5)$$

Suppose that the seller has chosen σ. Then, the game option reduces to an American option with payoff $I(., \sigma)$, whose initial price is given by $\sup_{\tau \in \mathcal{T}} \mathcal{E}^g_{0,\tau\wedge\sigma}[I(\tau, \sigma)]$. Set

$$Y(0) := \inf_{\sigma \in \mathcal{T}} \sup_{\tau \in \mathcal{T}} \mathcal{E}^g_{0,\tau\wedge\sigma}[I(\tau, \sigma)]. \quad (6)$$

called the **fair value** of the game option in the sequel.
$\rightarrow$ new game problem.

Generalized Dynkin games

Let ξ and $\zeta \in \mathcal{S}^2$ such that $\xi \leq \zeta$ and $\xi_T = \zeta_T$ a.s.
For each $\tau, \sigma \in \mathcal{T}_0$, let

$$I(\tau, \sigma) = \xi_\tau \mathbf{1}_{\tau \leq \sigma} + \zeta_\sigma \mathbf{1}_{\sigma < \tau}.$$

For $S \in \mathcal{T}_0$,

$$\overline{V}(S) := ess \inf_{\sigma \in \mathcal{T}_S} ess \sup_{\tau \in \mathcal{T}_S} \mathcal{E}_{\mathcal{S}, \tau \wedge \sigma}[\mathcal{I}(\tau, \sigma)]$$

$$\underline{V}(S) := ess \sup_{\tau \in \mathcal{T}_S} ess \inf_{\sigma \in \mathcal{T}_S} \mathcal{E}_{\mathcal{S}, \tau \wedge \sigma}[\mathcal{I}(\tau, \sigma)].$$

We clearly have the inequality $\underline{V}(S) \leq \overline{V}(S)$ a.s.

$$\overline{V}(S) := ess \inf_{\sigma \in \mathcal{T}_S} ess \sup_{\tau \in \mathcal{T}_S} \mathcal{E}_{\mathcal{S},\tau\wedge\sigma}[\mathcal{I}(\tau,\sigma)]$$

$$\underline{V}(S) := ess \sup_{\tau \in \mathcal{T}_S} ess \inf_{\sigma \in \mathcal{T}_S} \mathcal{E}_{\mathcal{S},\tau\wedge\sigma}[\mathcal{I}(\tau,\sigma)].$$

Definition

we say that the game is **fair** at time S if $\overline{V}(S) = \underline{V}(S)$ a.s.

Definition

Let $S \in \mathcal{T}_0$. A pair $(\tau^*, \sigma^*) \in \mathcal{T}_S^2$ is called an **S-saddle point** if $\forall$ $(\tau, \sigma) \in \mathcal{T}_S^2$, we have

$$\mathcal{E}_{S,\tau\wedge\sigma^*}[I(\tau,\sigma^*)] \leq \mathcal{E}_{S,\tau^*\wedge\sigma^*}[I(\tau^*,\sigma^*)] \leq \mathcal{E}_{S,\tau^*\wedge\sigma}[I(\tau^*,\sigma)] a.s.$$

Double barrier reflected BSDEs with jumps

Let ξ and $\zeta \in \mathcal{S}^2$ such that $\xi_t \leq \zeta_t$ and $\xi_T = \zeta_T$ a.s.

Definition

Solution: (Y, Z, k, A, A') in $\mathcal{S}^2 \times \boldsymbol{H}^2 \times \boldsymbol{H}^2_\nu \times (\mathcal{A}^2)^2$ such that

$$-dY_t = g(t, Y_t, Z_t, k_t)dt + dA_t - dA_t^{'} - Z_t dW_t - \int_{\mathbf{R}^*} k_t(u)\tilde{N}(dt, du);$$

$$Y_T = \xi_T, \tag{7}$$

$$\xi_t \leq Y_t \leq \zeta_t,\ 0 \leq t \leq T \text{ a.s.,}$$

$$\int_0^T (Y_t - \xi_t) dA_t^c = 0 \text{ a.s. and } \int_0^T (\zeta_t - Y_t) dA_t^{'c} = 0 \text{ a.s.} \tag{8}$$

$$\Delta A_\tau^d = \Delta A_\tau^d \mathbf{1}_{\{Y_{\tau^-} = \xi_{\tau^-}\}} \text{ and } \Delta A_\tau^{'d} = \Delta A_\tau^{'d} \mathbf{1}_{\{Y_{\tau^-} = \zeta_{\tau^-}\}} \text{ a.s. } \forall \tau \in \mathcal{T}_0 \text{ predictable}$$

$$dA_t \perp dA_t' \tag{9}$$

A particular classical case: g does not depend on y, z

Fix $S \in \mathcal{T}_0$.

$\forall\ \tau, \sigma \in \mathcal{T}_S$, define

$$I_S(\tau,\sigma) := \int_S^{\sigma\wedge\tau} g_s ds + \xi_\tau \mathbf{1}_{\{\tau\leq\sigma\}} + \zeta_\sigma \mathbf{1}_{\{\sigma<\tau\}}$$

We have

$$\overline{V}(S) = ess \inf_{\sigma\in\mathcal{T}_S} ess \sup_{\tau\in\mathcal{T}_S} \mathbf{E}[I_S(\tau,\sigma)|\mathcal{F}_S]$$

$$\underline{V}(S) = ess \sup_{\tau\in\mathcal{T}_S} ess \inf_{\sigma\in\mathcal{T}_S} \mathbf{E}[I_S(\tau,\sigma)|\mathcal{F}_S]$$

$\rightarrow$ **Classical Dynkin games (see e.g. Cvitanic and K. (1996), Hamadène)** They show that the value function of the **classical** Dynkin game coincides with the solution of the doubly reflected BSDE associated with the **driver** process g_t (which **does not depend on** y, z).

Recall that classicaly, we introduce

$$\tilde{\xi}^g_t := \xi_t - \mathbb{E}[\xi_T + \int_t^T g(s)ds | F_t], \quad \tilde{\zeta}^g_t := \zeta_t - \mathbb{E}[\xi_T + \int_t^T g(s)ds | F_t], \quad 0 \leq$$

so that $\tilde{\xi}^g_T = \tilde{\zeta}^g_T = 0$ a.s.
By results on classical Dynkin games , one can construct by using a recursive procedure two supermartingales J^g and J'^g, valued in $\mathbb{R}^+ \cup \{+\infty\}$ (see K.-Q.-C. 2013) which satisfy:

$$J^g = \mathcal{R}(J'^g + \tilde{\xi}) \quad J'^g = \mathcal{R}(J^g - \tilde{\zeta}).$$

Then, when J^g and J'^g are finite (which is the case when under Mokobodski's condition), then (see e.g. Cvitanic and K...)

$$\overline{Y}_t := J^g_t - J'^g_t + E[\xi_T + \int_t^T g(s)ds | \mathcal{F}_t]; \; 0 \leq t \leq T.$$

is solution of the doubly reflected BSDE associated with the **driver** process $g(s)$ (which **does not depend on** y, z).

Doubly reflected BSDEs with a general driver $g(t, y, z, k)$

Here the **driver** $g(t, y, z, k)$ **depends on** y, z.
Recall that under Mokobodski's condition,
the DRBSDE associated with general driver $g(t, y, z, k)$ admits a unique solution $(Y, Z, k, A, A') \in \mathcal{S}^2 \times \boldsymbol{H}^2 \times \boldsymbol{H}^2_\nu \times (\mathcal{A}^2)^2$.
Remark: In the **previous literature** (Cvitanic-K. ....), the authors have noted that the solution Y of the DRBSDE coincides with the value function of the previous **classical Dynkin game** with $g_s := g(s, Y_s, Z_s, k_s)$. Here, the gain is given by

$$I_S(\tau, \sigma) = \int_S^{\sigma \wedge \tau} g(u, Y_u, Z_u, k_u) du + \xi_\tau \mathbf{1}_{\{\tau \leq \sigma\}} + \zeta_\sigma \mathbf{1}_{\{\sigma < \tau\}}. \quad (10)$$

But it is **not so interesting** because the instantaneous reward process $g_s := g(s, Y_s, Z_s, k_s)$ **depends on the value function** Y of the associated Dynkin game **itself**.

Generalized Dynkin Game

(Here, $g(t, y, z, k)$ **depends on** y, z)

Definition: Let $S \in \mathcal{T}_0$. A pair $(\hat{\tau}, \hat{\sigma}) \in \mathcal{T}_S^2$ is an *S-saddle point* if $\forall$ $(\tau, \sigma) \in \mathcal{T}_S^2$, we have

$$\mathcal{E}_{S,\tau\wedge\hat{\sigma}}[I(\tau, \hat{\sigma})] \leq \mathcal{E}_{S,\hat{\tau}\wedge\hat{\sigma}}[I(\hat{\tau}, \hat{\sigma})] \leq \mathcal{E}_{S,\hat{\tau}\wedge\sigma}[I(\hat{\tau}, \sigma)] a.s.$$

The classical sufficient condition of "optimality" for the classical Dynkin game, based on J^g and J'^g (see Alario-N.et al. (1982)), is not appropriate to our case. Here, we have

Lemma (Sufficient condition of "optimality", Dum.-Que-Sul. 2013)

Let (Y, Z, k, A, A') be the solution of the DBBSDE .
Let $(\hat{\tau}, \hat{\sigma}) \in \mathcal{T}_S$.
Suppose $(Y_t, S \leq t \leq \hat{\tau})$ is an $\mathcal{E}$-submartingale and $(Y_t, S \leq t \leq \hat{\sigma})$ is an $\mathcal{E}$-supermartingale with $Y_{\hat{\tau}} = \xi_{\hat{\tau}}$ and $Y_{\hat{\sigma}} = \zeta_{\hat{\sigma}}$ a.s.
$\Rightarrow$ $(\hat{\tau}, \hat{\sigma})$ is a S-saddle point and

$$Y_S = \overline{V}(S) = \underline{V}(S) \ a.s.$$

Proof:

Let $\tau \in \mathcal{T}_S$. We want to show that for each $\tau \in \mathcal{T}_S$

$$Y_S \geq \mathcal{E}_{S,\tau\wedge\hat{\sigma}}(I(\tau,\hat{\sigma})) \text{ a.s.} \tag{11}$$

Since the process $(Y_t, S \leq t \leq \tau \wedge \hat{\sigma})$ is an $\mathcal{E}$ supermartingale,

$$Y_S \geq \mathcal{E}_{S,\tau\wedge\hat{\sigma}}(Y_{\tau\wedge\hat{\sigma}}). \tag{12}$$

Since $Y \geq \xi$ and $Y_{\hat{\sigma}} = \zeta_{\hat{\sigma}}$ a.s. , we have

$$Y_{\tau\wedge\hat{\sigma}} = Y_\tau \mathbf{1}_{\tau\leq\hat{\sigma}} + Y_{\hat{\sigma}} \mathbf{1}_{\hat{\sigma}<\tau} \geq \xi_\tau \mathbf{1}_{\tau\leq\hat{\sigma}} + \zeta_{\hat{\sigma}} \mathbf{1}_{\hat{\sigma}<\tau} = I(\tau,\hat{\sigma}).$$

By (12) and since $\mathcal{E}$ is increasing, we derive (11).
Similarly, for each $\sigma \in \mathcal{T}_S$:

$$Y_S \leq \mathcal{E}_{S,\hat{\tau}\wedge\sigma}(I(\hat{\tau},\sigma)) \quad \text{a.s.}$$

$\Rightarrow$ $(\hat{\tau},\hat{\sigma})$ is an S-saddle point and $Y_S = \overline{V}(S) = \underline{V}(S)$ a.s.

Theorem (Existence of S-saddle point, D-Q-S. 2013)

Let (Y, Z, k, A, A') be the solution of the DBBSDE.
Suppose that A, A' **are continuous** *(which is the case if ξ and $-\zeta$ are left-u.s.c. along s.t.).*
For each $S \in \mathcal{T}_0$, let

$$\sigma_S^* := \inf\{t \geq S,\ Y_t = \zeta_t\}; \quad \tau_S^* := \inf\{t \geq S,\ Y_t = \xi_t\},$$

$\Rightarrow$ *(τ_S^*, σ_S^*) is an S-saddle point for $Y_S = \overline{V}(S) = \underline{V}(S)$ a.s.*

proof: Since Y and ξ are cad, we have $Y_{\sigma_S^*} = \xi_{\sigma_S^*}$ and $Y_{\tau_S^*} = \xi_{\tau_S^*}$ a.s. Also, $Y_t > \xi_t$ for each $t \in [S, \tau_S^*[$. Hence, since Y is solution of the DBBSDE, A is constant on $[S, \tau_S^*]$ a.s.

proof:

- Since Y and ξ are cad, we have $Y_{\sigma_S^*} = \xi_{\sigma_S^*}$ and $Y_{\tau_S^*} = \xi_{\tau_S^*}$ a.s. Also, $Y_t > \xi_t$ for each $t \in [S, \tau_S^*[$. Hence, since Y is solution of the DBBSDE, A is constant on $[S, \tau_S^*]$ a.s.
- $\Rightarrow$ Y is an $\mathcal{E}$-submartingale on $[S, \tau_S^*]$.
- Similarly, Y is an $\mathcal{E}$-supermartingale on $[S, \sigma_S^*]$.
- By the Lemma, (τ_S^*, σ_S^*) is an S-saddle point and $Y_S = \overline{V}(S) = \underline{V}(S)$ a.s.

The main result

Here, A, A' are **not** supposed to be **continuous**. There does not a priori exist a saddle-point. However,

Theorem (Characterization, D.-Q.-S. 2013)

Let (Y, Z, k, A, A') be the solution of the doubly reflected BSDE associated with the **nonlinear** *driver $g(t, y, z, k)$.*
The **Generalized Dynkin game is fair** *and*

$$Y_S = \overline{V}(S) = \underline{V}(S) \quad a.s.$$

Sketch of the proof:

$$\tau_S^\varepsilon := \inf\{t \geq S,\ Y_t \leq \xi_t + \varepsilon\}.$$

$$\sigma_S^\varepsilon := \inf\{t \geq S,\ Y_t \geq \zeta_t - \varepsilon\}.$$

We first show that $A_{\tau_S^\varepsilon} = A_S$ a.s. and $A'_{\sigma_S^\varepsilon} = A'_S$ a.s.
We then derive that $(\tau_S^\varepsilon, \sigma_S^\varepsilon)$ is a $K\epsilon$-saddle point at time S and the desired result.

Application to game options in the market with constraints

- **Corollary** (Dumistrescu-Quenez-Sulem (2014)) The *fair value* of the game option satisfies

$$Y(0) = \inf_{\sigma \in \mathcal{T}} \sup_{\tau \in \mathcal{T}} \mathcal{E}^g_{0,\tau\wedge\sigma}(I(\tau,\sigma)) = \sup_{\tau \in \mathcal{T}} \inf_{\sigma \in \mathcal{T}} \mathcal{E}^g_{0,\tau\wedge\sigma}[I(\tau,\sigma)] = Y_0,$$

  where (Y, Z, K, A, A') is the unique solution in $\mathcal{S}^2 \times L^2(W) \times L^2(M) \times \mathcal{A}^2 \times \mathcal{A}^2$ of the doubly reflected BSDE with nonlinear driver $g(t, y, z, k)$.
- In the **particular case** when g is **linear** with respect to y, z, $\rightarrow$ Hamadène's result.
- **Definition**: for each initial wealth x, a *super-hedge* against the game option is a pair (σ, φ) of a s.t. $\sigma \in \mathcal{T}$ and a strategy φ such that

$$V_t^{x,\varphi} \geq \xi_t,\ 0 \leq t \leq \sigma \text{ and } \quad V_\sigma^{x,\varphi} \geq \zeta_\sigma \text{ a.s.}$$

- $\mathcal{A}(x) :=$ *set of all super-hedges associated with* x.
- The **super-hedging price** is defined by

$$u_0 := \inf\{x \in \mathbf{R},\ \exists(\sigma,\varphi) \in \mathcal{A}(x)\}.$$

Theorem (Dum-Que-Sul 2015):

- Let (Y, Z, K, A, A') is the solution of the DRBSDE. **Suppose** A' **is continuous** (satisfied if ζ is left lower-s.c. along s.t.)
- Then, **super-hedging price** $=$ **fair value** of the game option, that is
$$u_0 = Y_0.$$
- Let $\sigma^* := \inf\{t \geq 0, \; Y_t = \zeta_t\}$ and $\varphi^* := \Phi(Z, K)$ (defined as before).
  Then, (σ^*, φ^*) belongs to $\mathcal{A}(Y_0)$.

Rem 1: Under these assumptions, there **does not a priori exist** τ^* such that (τ^*, σ^*) is a saddle point.
Rem 2: If A' is not continuous, then, generally, $u_0 \neq Y_0$.

A mixed game problem with nonlinear expectations

- **Two actions**: stopping times and controls $(u, v) \in \mathcal{U} \times \mathcal{V}$.
- A classical mixed game problem (Hamadène, Lepeltier) :
  The criterium is

$$E_{Q^{u,v}}\left[\int_S^{\tau\wedge\sigma} c(t, u_t, v_t)dt + I(\tau, \sigma)|\mathcal{F}_S\right],$$

  with $Q^{u,v}$ the probability with density $Z_T^{u,v}$ /

$$dZ_t^{u,v} = Z_t^{u,v}[\beta(t, u_t, v_t)dW_t + \int_{\mathbf{R}^*} \gamma(t, u_t, v_t, e)\tilde{N}(dt, de)];\ Z_0^{u,v} = 1.$$

- First player: chooses $(u, \tau) \in \mathcal{U} \times \mathcal{T}_S$ and aims to maximize the criterium
  Second player: chooses $(v, \sigma) \in \mathcal{V} \times \mathcal{T}_S$ and aims to minimize the criterium.

Generalized mixed game problem

Let $(g^{u,v}; (u,v) \in \mathcal{U} \times \mathcal{V})$ be a family of Lipschitz drivers / **A.1** .
Let $S \in \mathcal{T}_0$. For each $(u, \tau, v, \sigma) \in \mathcal{U} \times \mathcal{T}_S \times \mathcal{V} \times \mathcal{T}_S$, the *criterium* at time S is :

$$\mathcal{E}^{u,v}_{S,\tau\wedge\sigma}(I(\tau,\sigma)),$$

where $\mathcal{E}^{u,v} = g^{u,v}$-conditional expectation.
For each $S \in \mathcal{T}_0$,

$$\overline{V}(S) := ess \inf_{v\in\mathcal{V},\sigma\in\mathcal{T}_S} ess \sup_{u\in\mathcal{U},\tau\in\mathcal{T}_S} \mathcal{E}^{u,v}_{S,\tau\wedge\sigma}(I(\tau,\sigma)); \qquad (13)$$

$$\underline{V}(S) := ess \sup_{u\in\mathcal{U},\tau\in\mathcal{T}_S} ess \inf_{v\in\mathcal{V},\sigma\in\mathcal{T}_S} \mathcal{E}^{u,v}_{S,\tau\wedge\sigma}(I(\tau,\sigma)). \qquad (14)$$

Definition

Let $S \in \mathcal{T}_0$. A quadruple $(\overline{u}, \overline{\tau}, \overline{v}, \overline{\sigma}) \in \mathcal{U} \times \mathcal{T}_S \times \mathcal{V} \times \mathcal{T}_S$ is called an *S-saddle point* if for each (u, τ, v, σ) we have

$$\mathcal{E}^{u,\overline{v}}_{S,\tau\wedge\overline{\sigma}}(I(\tau,\overline{\sigma})) \leq \mathcal{E}^{\overline{u},\overline{v}}_{S,\overline{\tau}\wedge\overline{\sigma}}(I(\overline{\tau}\wedge\overline{\sigma})) \leq \mathcal{E}^{\overline{u},v}_{S,\overline{\tau}\wedge\sigma}(I(\overline{\tau},\sigma)) \quad \text{a.s.}$$

Existence of saddle points for the mixed game problem

Theorem (Dum.-Que-Sul. 2013)

Suppose ξ and ζ are left u.s.c. along stopping times + Mokobodski's condition.
Suppose that $\exists$ $\overline{u} \in \mathcal{U}$ and $\overline{v} \in \mathcal{V}$ such that for each $(u, v) \in \mathcal{U} \times \mathcal{V}$,

$$g^{u,\overline{v}}(t, Y_t, Z_t, k_t) \leq g^{\overline{u},\overline{v}}(t, Y_t, Z_t, k_t) \leq g^{\overline{u},v}(t, Y_t, Z_t, k_t) \quad dt \otimes dP \ a.s. \ ,$$

where (Y, Z, k, A, A') is the solution of the DBBSDE associated with driver $g^{\overline{u},\overline{v}}$. Let

$$\tau_S^* := \inf\{t \geq S : Y_t = \xi_t\} \quad ; \quad \sigma_S^* := \inf\{t \geq S : Y_t = \zeta_t\}.$$

The quadruple $(\overline{u}, \tau_S^*, \overline{v}, \sigma_S^*)$ is then an **S-saddle point** *and $Y_S = \underline{V}(S) = \overline{V}(S)$ a.s.*

The generalized mixed game problem **is fair**.

Suppose ξ and ζ are **not left u.s.c.** along stopping times. We have

Theorem (Dum.-Que-Sul. 2013)

Suppose that $\exists\ \overline{u} \in \mathcal{U}$ *and* $\overline{v} \in \mathcal{V}$ *such that for each* $(u, v) \in \mathcal{U} \times \mathcal{V}$,

$$g^{u,\overline{v}}(t, Y_t, Z_t, k_t) \leq g^{\overline{u},\overline{v}}(t, Y_t, Z_t, k_t) \leq g^{\overline{u},v}(t, Y_t, Z_t, k_t) \quad dt \otimes dP \ a.s. \ ,$$

where (Y, Z, k, A, A') *is the solution of the DBBSDE associated with driver* $g^{\overline{u},\overline{v}}$.
Then, the **generalized mixed game** *problem* **is fair**. *and* $Y_S = \underline{V}(S) = \overline{V}(S)$ *a.s.*

There does not necessarily exist a saddle point.

Application:

Let U, V be compact Polish spaces.
Let $F : [0, T] \times \Omega \times U \times V \times \mathbf{R}^2 \times L^2_\nu \to \mathbf{R}$,
$(t, \omega, u, v, y, z, k) \mapsto F(t, \omega, u, v, y, z, k)$, supposed to be measurable with respect to $\mathcal{P} \otimes \mathcal{B}(U) \otimes \mathcal{B}(V) \otimes \mathcal{B}(\mathbf{R}^2) \otimes \mathcal{B}(L^2_\nu)$, continuous, concave (resp. convex) with respect to u (resp. v), and uniformly Lipchitz with respect to (y, z, k). Suppose that $F(t, \omega, u, v, 0, 0, 0)$ is uniformly bounded.

Let $\mathcal{U}$ (resp. $\mathcal{V}$) be the set of predictable processes valued in U (resp. V). For each $(u, v) \in \mathcal{U} \times \mathcal{V}$, let $g^{u,v}$ be the driver defined by

$$g^{u,v}(t, \omega, y, z, k) := F(t, \omega, u_t(\omega), v_t(\omega), y, z, k).$$

Define for each (t, ω, y, z, k)

$$g(t, \omega, y, z, k) := \sup_{u \in U} \inf_{v \in V} F(t, \omega, u, v, y, z, k). \tag{15}$$

g is a Lipschitz driver.

Let $(Y, Z, k, A, A') \in \mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\nu \times (\mathcal{A}^2)^2$ be the solution of the DRBSDE associated with g.

By classical convex analysis, and then by applying a selection theorem, we get that $\exists$ predictable process $(u^*, v^*) \in \mathcal{U} \times \mathcal{V}$ such that $dt \otimes dP$ a.s., for all $(u, v) \in \mathcal{U} \times \mathcal{V}$ we have $dt \otimes dP$ a.s.:

$$F(t, u_t, v_t^*, Z_t, k_t) \leq F(t, u_t^*, v_t^*, Y_t, Z_t, k_t) \leq F(t, u_t^*, v_t, Y_t, Z_t, k_t)$$

and $g(t, Y_t, Z_t, k_t) := F(t, u_t^*, v_t^*, Y_t, Z_t, k_t)$. Hence, Assumption (12) is satisfied. By the above Theorems, we derive :

Proposition (i) The **generalized mixed game** problem **is fair**.
Let Y be the solution of the DRBSDE associated with obstacles ξ, ζ and the driver g defined by (15).
For each stopping time $S \in \mathcal{T}_0$, we have $Y_S = \overline{V}(S) = \underline{V}(S)$ a.s.

Proposition

(i) The **generalized mixed game** problem (associated with the map $F(t,u,v,y,z,k)$) **is fair**. Let Y be the solution of the DRBSDE associated with obstacles ξ, ζ and the driver g defined by (15).

For each stopping time $S \in \mathcal{T}_0$, we have $Y_S = \overline{V}(S) = \underline{V}(S)$ a.s.

(ii) **Suppose now that ξ and $-\zeta$ are l.u.s.c. along s.t.** Set

$$\tau_S^* := \inf\{t \geq S : Y_t = \xi_t\} \quad ; \quad \sigma_S^* := \inf\{t \geq S : Y_t = \zeta_t\}.$$

The quadruple $(u^*, \tau_S^*, v^*, \sigma_S^*)$ is then an S-saddle point for this mixed game problem.

Other useful applications of our main result

- From the characterization theorem, we easily derive a **comparison theorem** for Doubly RBSDEs, which generalizes the one obtained by Crepey-Matoussi (2008).
- We also derive **new a priori estimates** for Doubly RBSDEs with **universal constants**.
  Remark: Under some **additional assumptions** on the barriers, Crepey-Matoussi (2008) have proved a priori estimates (but with **non** universal constants).
- **These estimates are an efficient tool** to study the **Markovian** case for DRBSDEs (see Dum-Quen-Sul (2013)).
- and also the **Markovian** case with uncertainty, that is a mixed generalized DG (see Dum-Quen-Sul (2015)).