Optimal contracting for a system of interacting Agents

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Work in progress joint with Romuald Élie

London-Paris Bachelier Workshop, September 25th 2015, London

Motivation

- Large economic literature on contract design in order to revisit equilibrium theory by incorporating incentives and asymmetry of information ;
- Optimal contracting between a Principal and an Agent ;
- Mainly static or discrete time problems : *Spear & Srivastava*, *Salanié*, *Tirole*, *Laffont*, *Martimort*, *Radner* $\Longrightarrow$ Limited computations ;
- Extension to continuous time models : *Holmstrom & Milgrom*, *Schättler*, *Williams*, *Sung*, *Sannikov*, *Cvitanić & Zhang*,... ;
- More explicit solutions and strong connexion with the theory of BSDEs. ;

Revisiting the Holmstrom and Milgrom moral hazard problem

- Controlled output process (such as a production good, or cash flows ...)

$$dX_t^a = a_t dt + \sigma dB_t$$

- The control a is s.t. $\mathcal{E}(\int_0^T a_t/\sigma dB_t)$ is unif. integrable ;

- Weak formulation : choice of a probability $\mathbb{P}^a$ with Brownian Motion B^a

$$dX_t \;=\; a_t dt + \sigma dB_t^a$$

- The Agent observes B^a whereas the Principal only observes X ;

- The Agent chooses a control $(a_t)_t$ and receives a payment ξ at time T. He solves

$$\sup_a \mathbb{E}^{\mathbb{P}^a}\left[U_A\left(\xi - \int_0^T k(a_t)dt\right)\right]$$

- The Principal chooses and pays the terminal payment ξ and solves

$$\sup_\xi \mathbb{E}^{\mathbb{P}^{a^\star(\xi)}}\left[U_P(X_T - \xi)\right]$$

- U_A and U_P are exponential utility functions with risk aversions R_A and R_P.

Solving the Agent's problem

- Consider a given $\mathcal{F}_T^X$-measurable payment contract ξ
- The problem of the Agent rewrites as follows

$$U_0^A = \sup_a \mathbb{E}^{\mathbb{P}^a}\left[U_A\left(\xi - \int_0^T k(a_s)ds\right)\right]$$

- The dynamic version at time t is

$$U_t^A(a) = \operatorname*{ess\,sup}_{a' \, on[t,T]} J_t^{a,a'}, \text{ where } J_t^{a,a'} = \mathbb{E}^{\mathbb{P}^{a'}}\left[U_A\left(\xi - \int_t^T k(a_s')ds\right)\middle| \mathcal{F}_t^B\right]$$

- $e^{R_A \int_0^t k(a_s)ds} J_t^{a,a'}$ is a $\mathbb{P}^{a'}$- martingale
- Introduce the log target process $Y^{a,a'} := \frac{-ln(J_\cdot^{a,a'})}{R_A}$
- Ito's formula and Girsanov Theorem imply that $Y^{a,a'}$ solves the BSDE

$$Y_t^{a,a'} = \xi + \int_t^T f(a', Z_s^{a,a'})ds - \int_t^T Z_s^{a,a'} \sigma dB_s,$$

with the family of drivers $f : (a', z) \mapsto -\frac{R_A}{2}\sigma^2 z^2 + a'z - k(a')$.

Solving the Agent's problem

- We want to maximize $J^{a,a'}$ or similarly $Y^{a,a'}$ over a'
- Comparison for BSDEs $\Longrightarrow$ take $f^* := \sup_{a'} f(a', .)$. Original approach of El Karoui & Quenez for the link between stochastic control and BSDEs.
- Optimal control a^* which maximizes the driver.
- The class of admissible contracts relies on integrability conditions on ξ for the BSDE to be well posed

Back to the Principal's problem

- For a given terminal payment ξ, the log expected utility Y of the Agent is

$$Y_t = \xi + \int_t^T f^*(Z_s^*)ds - \int_t^T Z_s^* \sigma dB_s$$

- The participation constraint of the Agent requires $Y_0 \geq y_0$

  $\Longrightarrow$ Consider terminal payment ξ of the form

$$\xi = y_0 + \int_t^T f^*(Z_s^*)ds - \int_t^T Z_s^* \sigma dB_s$$

- The problem of the Principal is

$$\sup_{\xi} \mathbb{E}^{P^{a^*(Z^*)}}[U_P(X_T - \xi)]$$

- Maximizing over Z^*, it rewrites

$$\sup_{Z^*} \mathbb{E}^{P^{a^*(Z^*)}} \left[\mathcal{E}\left(-R_p \int_0^T \sigma(1 - Z_s^*) dB_s^{a^*(Z^*)} \right) U_p \left(\int_0^T \beta(Z_s^*) ds \right) \right]$$

  with $\beta : z \mapsto a^*(z) + f^*(z) - za^*(z) - \frac{R_p}{2}\sigma^2(1-z)^2$

- Optimal $Z_\cdot$ is deterministic and given by a constant pointwise sup z^* of β

Solution to the Holstrom and Milgrom moral hazard problem

- The optimal contract is linear in the output process X

$$\xi_T^* = y_0 - Tf^*(z^*) + z^*(X_T - Ta^*(z^*))$$

- It provides exactly his utility of reservation to the Agent

- Example : quadratic cost $k : a \mapsto \frac{ka^2}{2}$

  Optimal proportion of X in the contract

$$z^* = \frac{R_P + \frac{1}{k\sigma^2}}{R_A + R_P + \frac{1}{k\sigma^2}}$$

  Higher than in the first best case :

$$\frac{R_P}{R_A + R_P}$$

Some extensions and related literature

- Thanks to the BSDE approach, the Principal problem actually becomes a classical stochastic control problem.
- In general, one has to interpret the value of the contract as the terminal value of a new state variable, which is nothing more than a transformation of the continuation of the Agent.
- One can then write the HJB equation associated to the Principal problem.
- This approach allows to tackle problems where the Agent also controls the volatility of the output, Cvitanić, P., Touzi (2015).

A Principal wishes to hire N Agents

- A Principal requires to handle an N-dimensional output process X

$$dX_t = \Sigma_t dB_t, \text{ with } \Sigma \text{ bounded and invertible.}$$

- She wishes to hire N Agents
- Each Agent will be assigned to one project but he can choose to impact (positively or negatively) any project.
- The control process for Agent j is a vector a^j, such that a^{ij} is the control used by Agent j in order to impact the project i.
- The controlled process X is given by

$$dX_t = b(t, a_t)dt + \Sigma_t dB_t^a,$$

where B^a is a BM under $\mathbb{P}^a$ defined by $\frac{d\mathbb{P}^a}{d\mathbb{P}} = \mathcal{E}\left(-\int_0^T b(s, a_s) \cdot \Sigma_s^{-1} dB_s\right)$.

Agents with relative performance concerns

The satisfaction of each Agent comes from

- His terminal payment ξ^i ;
- How well his project performed in comparison to the other Agents

The Agent optimization problem is

$$\sup_{a^{:,i}} \mathbb{E}^{\mathbb{P}^a}\left[U_i^A\left(\xi^i + \Gamma_i(X_T) - \int_0^T k^i(s, a_s^{:,i})ds\right)\right]$$

- Γ_i is a competition index for Agent i.
- For instance, similar to relative performance concerns as in e.g. *Espinosa-Touzi*

$$\Gamma_i(X_T) := \gamma_i(X_T^i - \bar{X}_T^{-i}).$$

Solving the first best problem : No moral hazard

- First best problem
  $\Longrightarrow$ The Principal chooses the payments and the actions of the Agents.

- We denote

$$\bar{\gamma}^{-i} := \frac{1}{N-1}\sum_{j\neq i}\gamma_j \qquad \text{and} \qquad \frac{1}{\overline{R}_A} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{R_A^i}$$

Theorem

Given the reservation utilities (U_0^i) of the Agents, the optimal first best payment is :

$$\xi_{FB}^i := \frac{R_P\overline{R}_A}{R_A^i(\overline{R}_A+NR_P)}(\mathbf{1}_N+\gamma-\overline{\gamma}^-)\cdot X_T-\gamma_i\left(X_T^i-\overline{X}_T^{-i}\right)+C_i,$$

where the optimal action a_t^* is any minimizer of the map

$$a \longmapsto (\overline{\gamma}^- - \gamma - \mathbf{1}_N)\cdot b(t,a)+\mathbf{1}_N\cdot k(t,a)\ .$$

Economic interpretation

- Each Agent is penalized with the amount $-\gamma^i(X_T^i - \bar{X}_T^{-i})$, so as to suppress the appetence for competition of the Agents.
- Moreover, each Agent is paid a positive part of each projects, the percentage depending on the risk aversion of the Agent, and of the universal vector

$$\frac{R_P \overline{R}_A}{\overline{R}_A + N R_P}(\mathbf{1}_N + \gamma - \overline{\gamma}^-).$$

- If an Agent is particularly competitive, then any Agent will receive a large part of his project
- If an Agent is not very competitive, other Agents have incitation to reduce the value of his project as much as possible.

Linear drift and quadratic cost functions :

$$b(t,a) := \begin{pmatrix} a^{11} - a^{12} \\ a^{22} - a^{21} \end{pmatrix} \quad \text{and} \quad k(t,a) := \begin{pmatrix} \frac{k^{11}}{2}\left|a^{11}\right|^2 + \frac{k^{21}}{2}\left|a^{21}\right|^2 \\ \frac{k^{22}}{2}\left|a^{22}\right|^2 + \frac{k^{12}}{2}\left|a^{12}\right|^2 \end{pmatrix}$$

$\Longrightarrow$ Optimal actions of the two Agents are

Agent 1 : $\quad a^{11} = \dfrac{1+\gamma_1-\gamma_2}{k^{11}} \qquad a^{21} = -\dfrac{1+\gamma_2-\gamma_1}{k^{21}}$

Agent 2 : $\quad a^{12} = \dfrac{1+\gamma_2-\gamma_1}{k^{12}} \qquad a^{22} = -\dfrac{1+\gamma_1-\gamma_2}{k^{22}}$

If Agent 1 is much more competitive than Agent 2 :

- Agent 1 will work towards his project and will also work to decrease the value of the project of Agent 2 ;
- Agent 2 will work to decrease the value of his own project and to increase the value of the project of Agent 1.

Optimal recruitment scheme

- Let consider Agents with similar reservation utilities.

  What is the optimal competition scheme between Agents from the Principal viewpoint ?

- Maximizing the value function of the Principal boils down to minimizing

$$g : (\gamma_1, \gamma_2) \mapsto (1+\gamma_1-\gamma_2)^2 \alpha_1 + (1+\gamma_2-\gamma_1)^2 \alpha_2,$$

  where

$$\alpha_1 := \frac{R_P \overline{R}_A}{\overline{R}_A + 2R_P}\sigma_1^2 - \left(\frac{1}{k^{11}} + \frac{1}{k^{22}}\right), \ \alpha_2 := \frac{R_P \overline{R}_A}{\overline{R}_A + 2R_P}\sigma_2^2 - \left(\frac{1}{k^{12}} + \frac{1}{k^{21}}\right).$$

- For low working costs, $\alpha_1 + \alpha_2 \leq 0$

  The Principal would like to hire Agents with $|\gamma_1 - \gamma_2| \longrightarrow +\infty$

- For high working costs, $\alpha_1 + \alpha_2 > 0$,

  The Principal wants to hire Agents with $\gamma_1 - \gamma_2 = \dfrac{\alpha_2 - \alpha_1}{\alpha_1 + \alpha_2}$.

- More competitive Agents must work on less volatile projects.

Moral hazard with multiple Agents

- We now turn to the second best/moral hazard problem ;
- The Principal can not observe the actions of the Agents and only controls the salary ξ that he offers ;
- Similar ideas as in the BSDE scheme of proof derived for the Principal - unique Agent case ;
- Stackelberg equilibrium between the Principal and the system of Agents ;
- Nash equilibrium between all the interacting Agents.

Identifying the best reaction functions

- Let the terminal payment ξ be given
- Consider Agent i given the actions a^{-i} of the others

$$U_0^i(a^{-i},\xi^i) := \sup_{a\in\mathcal{A}^i(a^{-i})} \mathbb{E}^{\mathbb{P}^{a\otimes_i a^{-i}}}\left[U_A^i\left(\xi^i+\gamma_i\left(X_T^i-\overline{X}_T^{-i}\right)-\int_0^T k^i(s,a_s)ds\right)\right].$$

- As in the unique Agent case, this leads to the consideration of BSDEs for the log target process.

$$Y_t^{i,a^{-i},\xi^i} = \xi^i + \Gamma_i(X_T) + \int_t^T \tilde{f}^{i,a^{-i}}\left(s, Z_s^{i,a^{-i},\xi^i}, a\right) ds - \int_t^T Z_s^{i,a^{-i},\xi^i}\cdot\Sigma_s dW_s$$

with $\tilde{f}^{i,a^{-i}}(t,\omega,z,a) := -\dfrac{R_A^i}{2}\|\Sigma(t)z\|^2 + b(t, a\otimes_i a_t^{-i}(\omega))\cdot z - k^i(t,a_t).$

- Consider the maximal solution of the BSDE, if it exists.
- Hoping for comparison results for the BSDE, introduce

$$f^{i,a^{-i}}(t,\omega,z) := \sup_{a\in\mathcal{A}^i(a^{-i})} \tilde{f}^{i,a^{-i}}(t,\omega,z,a)$$

Theorem

There is a one-to-one correspondence between

- (i) *a Nash equilibrium* $a^*(\xi) \in \mathcal{A}$ *such that for any* $i = 1, \ldots, N$, *there is some* $p > 1$ *such that some martingale satisfies the reverse Holder inequality of order* p *for* $\mathbb{P}^{a^*(\xi)}$.
- (ii) *a solution* (Y, Z) *to the BSDE*

$$Y_t^\xi = \xi + \Gamma(X_T) + \int_t^T f(s, Z_s^\xi, X_s)ds - \int_t^T (Z_s^\xi)^\top \Sigma_s dW_s,$$

such that in addition $Z \in \mathbb{H}^2_{\mathrm{BMO}}(\mathbb{P}, \mathcal{M}_N(\mathbb{R}))$.

The correspondence is given by, for any $i = 1, \ldots, N$

$$({a^*}_s(\xi))^{:,i} \in \underset{a \in \mathcal{A}^i((a^*)^{:,-i})}{\operatorname{argmax}} \left\{ \sum_{j=1}^N b^j(s, (a \otimes_i ({a^*}_s)^{:,-i}(s, Z_s))^{:,j}) Z_s^{ij} - k^i(s, a_s) \right\},$$

Identifying the admissible contracts

Finding a Nash equilibria reduces to solving the multidimensional quadratic BSDE.

- For small bounded terminal condition ξ, results of *Tevzadze*.
- counter examples in general, e.g. *Frei and Dos Reis*.
- particular structures : *Cheredito & Nam, Kramkov & Pulido, Hu & Tang, Jamneshan et al., Luo & Tangpi, Kardaras, Xing & Žitković*

The Principal requires to offer a terminal payment ξ which produces a Nash equilibrium for the system of Agents.

$$\mathrm{NA}(\xi) := \{\text{Nash equilibria associated to } \xi \text{ satisfying the reverse Hölder cond.}\}$$

$$\mathrm{NAI}(\xi) := \{a \in \mathrm{NA}(\xi),\ a \succeq b,\ \text{for any } b \in \mathrm{NA}(\xi)\}.$$

Admissible contracts for the second best problem :

$$\mathcal{C}^{SB} := \left\{\xi \in \mathcal{C}^{FB},\ \mathrm{NAI}(\xi) \text{ is non-empty}\right\}.$$

- The Principal problem is

$$\sup_{\xi \in \mathcal{C}^{SB}} \sup_{a \in \mathrm{NAI}(\xi)} \mathbb{E}^{\mathbb{P}^a}\left[-e^{-R_P(X_T-\xi)\cdot \mathbf{1}_N} - \sum_{i=1}^{N} \rho_i e^{-R_A^i\left(\xi^i+\gamma_i\left(X_T^i-\overline{X}_T^{-i}\right)-\int_0^T k^i(s,a_s^{:,i})ds\right)}\right]$$

- The $(\rho_i)_i$ are the Lagrange multipliers of the participation constraints.
- For any $\xi \in \mathcal{C}^{SB}$, there is a pair $(Y_0^\xi, Z^\xi) \in \mathbb{R}^N \times \mathbb{H}^2_{\mathrm{BMO}}(\mathbb{P}, \mathcal{M}_N(\mathbb{R}))$ s. t.

$$\xi = Y_0^\xi - \Gamma(X_T) - \int_0^T f(s, Z_s^\xi, X_s)ds + \int_0^T Z_s^\xi \cdot \Sigma_s dW_s, \ \ a.s. \qquad (1)$$

- Optimization over the process Z^ξ.
- We know the actions a^* of the system of Agents in response to a payment $(Z_t^\xi)_t$.
- Optimal deterministic process Z^*.

The optimal contract for the moral hazard problem

Theorem

An optimal contract $\xi_{SB} \in \mathcal{C}^{SB}$ with reservation utilities $(U_0^i)_{1\leq i\leq N}$, is given by

$$\xi_{SB}^i := -\frac{1}{R_A^i}\log(-U_0^i) - \gamma_i(X_T^i - \overline{X}_T^{-i}) + \left(\int_0^T z_s^* dX_s\right)^i + C_i,$$

where z_t^* is any deterministic maximizer of the map

$$z \longmapsto \left(\mathbf{1}_N + \gamma - \overline{\gamma}^-\right)\cdot b(t, a_t^*(z)) - k(t, a_t^*(z))\cdot \mathbf{1}_N$$
$$- \sum_{i=1}^N \frac{R_A^i}{2}\|\Sigma(t)z^{:,i}\|^2 - \frac{R_P}{2}\|\Sigma\left(z^\top \mathbf{1}_N + \mathbf{1}_N + \gamma - \overline{\gamma}^-\right)\|^2.$$

Properties of the optimal contract

- z^* deterministic implies existence and uniqueness of solution for the optimal BSDE.
- For non-time dependent cost and drift function, the optimal contract is linear of the form
$$(\xi^*)^i(a^*) = C_i + z^* \cdot X_T - \gamma_i \left(X_T^i - \overline{X}_T^{-i} \right),$$
for some constant C_i.
- Each Agent gets his utility of reservation.
- Each Agent is paid a fixed part of each project.
- Each Agent is penalized with the amount $-\gamma^i(X_T^i - \bar{X}_T^{-i})$, so as to suppress the appetence for competition of the Agents.
- For the linear drift/quadratic cost case, we obtain more explicit formulae where in particular
$$a^*(z) = \begin{pmatrix} \frac{z^{11}}{k^{11}} & \frac{z^{12}}{k^{12}} \\ \frac{z^{21}}{k^{21}} & \frac{z^{22}}{k^{22}} \end{pmatrix}.$$
and z^* maximizes a 4-dimensional linear-quadratic function.

More general framework

- More general dynamics/drift for the controlled output

$$dX_t = b(t, a_t, X_t)dt + \Sigma_t dB^a_t$$

- Cost function $k(t, a_t, X_t)$;
- More general performance concerns :

$$\gamma^i(X^i_T - \bar{X}^{-i}_T), \text{ replaced by } \Gamma^i(X_T),$$

  with any linear growth function Γ^i.

- Same resolution for the Agent problem ;
- HJB characterization for the Principal problem in the second-best problem, and BSDE in the first best (explicit solution for quadratic costs).
- Recovers in particular the framework of *Goukasian & Wan* with relative payments concerns.

Thank you for your attention !