Hybrid scheme for Brownian semistationary processes

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Introduction

Hybrid scheme

Application to rough Bergomi model

Rough processes

- We are interested in efficient simulation methods for rough processes.
- Here, by “rough” we mean that the trajectories are rougher that those of Brownian motion. (*roughly* speaking. . .)
- Based on some empirical properties of realized volatility and implied volatility surfaces, Gatheral et al. (2014) have suggested that “volatility is rough”.
- Bayer et al. (2015) have introduced an option pricing model with rough volatility — the so-called rough Bergomi model.
- Rough processes are also useful in the modeling of electricity spot prices (Barndorff-Nielsen et al. 2013; Bennedsen 2015).

Brownian semistationary processes

Definition (Barndorff-Nielsen and Schmiegel, 2007)

Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in\mathbb{R}}, \mathbf{P})$ be a filtered probability space supporting a Brownian motion $\{W(t)\}_{t\in\mathbb{R}}$.

A Brownian semistationary ($\mathcal{BSS}$) process $\{X(t)\}_{t\in\mathbb{R}}$ is defined by

$$X(t) := \int_{-\infty}^{t} g(t-s)\sigma(s)\mathrm{d}W(s),$$

where

- $g : (0,\infty) \to [0,\infty)$ is a square-integrable kernel function,
- $\{\sigma(t)\}_{t\in\mathbb{R}}$ is an adapted covariance-stationary volatility process with locally bounded trajectories.

The role of the kernel function

The kernel function g strongly influences the behavior of the $\mathcal{BSS}$ process X:

- The behavior of g near zero influence the fine properties — such as roughness — of X.
- The asymptotics of g near infinity determine long-term behavior of X.

We consider kernel functions that satisfy:

$$g(x) \propto x^{\alpha}, \quad \text{when } x \text{ is near zero,}$$

for some $\alpha \in \left(-\frac{1}{2}, \frac{1}{2}\right) \setminus \{0\}$.

Key assumptions

Assumption

- For some $\alpha \in \left(-\frac{1}{2}, \frac{1}{2}\right) \setminus \{0\}$,

$$g(x) = x^{\alpha} L_g(x), \quad x \in (0,1],$$

where $L_g : (0,1] \to [0,\infty)$ is C^1, slowly varying at 0 ▸ Definition and bounded away from 0. Moreover, there exists a constant $C > 0$ such that the derivative L_g' of L_g satisfies

$$|L_g'(x)| \leq C(1 + x^{-1}), \quad x \in (0,1].$$

Key assumptions

Assumption

II The function g is C^1 on $(0, \infty)$, so that its derivative g' is ultimately monotonic and satisfies $\int_1^\infty g'(x)^2 dx < \infty$.

III For some $\beta \in \left(-\infty, -\frac{1}{2}\right)$,

$$g(x) = \mathcal{O}(x^\beta), \quad x \to \infty.$$

Example

The function

$$g(x) = x^\alpha e^{-\lambda x}, \quad x \in (0, \infty),$$

for any $\alpha \in \left(-\frac{1}{2}, \frac{1}{2}\right) \setminus \{0\}$ and $\lambda > 0$ satisfies these assumptions.

Stationarity and regularity of trajectories

Proposition

1. *The process X is centered and covariance stationary.*
2. *For any $t \in \mathbb{R}$,*

$$\mathbf{E}[|X(s) - X(t)|^2] \sim \mathbf{E}[\sigma(0)^2] C_\alpha |s-t|^{2\alpha+1} L_g(|s-t|)^2$$

as $s \to t$, where $C_\alpha = \frac{1}{2\alpha+1} + \int_0^\infty \left((y+1)^\alpha - y^\alpha\right)^2 dy$.

3. *The process X has a modification with locally ϕ-Hölder continuous trajectories for any $\phi \in \left(0, \alpha + \frac{1}{2}\right)$.*

Remark

We refer to α as the roughness parameter of X.

Simulation of $\mathcal{BSS}$ processes

We are interested in simulating discrete observations of the process

$$X(t) := \int_{-\infty}^{t} g(t-s)\sigma(s)\mathrm{d}W(s), \quad t \in \mathbb{R}.$$

If σ were deterministic, then X would be centered and Gaussian, making exact simulation possible.

- Computationally, exact simulation can be costly.
- Numerical evaluation of the covariance function of X might not be easy when g is singular, $\alpha < 0$.
- Moreover, this approach does not generalize to the case where σ is stochastic.

Thus, approximate simulation schemes seem unavoidable.

Approximation by Riemann sums

To approximate $X(t)$, an obvious method would be to use Riemann sums:

$$X(t) = \sum_{k=1}^{\infty} \int_{t-\frac{k}{n}}^{t-\frac{k}{n}+\frac{1}{n}} g(t-s)\sigma(s)\mathrm{d}W(s)$$
$$\approx \sum_{k=1}^{N_n} g\left(\frac{k}{n}\right)\sigma\left(t-\frac{k}{n}\right)\left(W\left(t-\frac{k}{n}+\frac{1}{n}\right)-W\left(t-\frac{k}{n}\right)\right),$$

where $N_n \to \infty$ as $n \to \infty$.

- This corresponds to approximating g by a step function.
- The scheme can be very inaccurate when g is singular, $\alpha < 0$.
- The first summands are the problematic ones, as g is evaluated near zero therein.

Hybrid scheme

We replace the first $\kappa \geqslant 1$ summands by random variables that provide a better approximation.

We use for $k = 1, \dots, \kappa$,

$$g(t-s) \approx (t-s)^{\alpha} L_g\left(\frac{k}{n}\right), \quad t-s \in \left[\frac{k-1}{n}, \frac{k}{n}\right] \setminus \{0\},$$

motivated by the properties slowly varying functions, and define

$$\check{X}_n(t) := \sum_{k=1}^{\kappa} L_g\left(\frac{k}{n}\right) \sigma\left(t - \frac{k}{n}\right) \int_{t-\frac{k}{n}}^{t-\frac{k}{n}+\frac{1}{n}} (t-s)^{\alpha} dW(s).$$

Hybrid scheme

We adopt the remaining summands from the Riemann sum, but we allow the point at which g is evaluated to be choosen freely within each discretization cell.

We define

$$\hat{X}_n(t) := \sum_{k=\kappa+1}^{N_n} g\left(\frac{b_k}{n}\right)\sigma\left(t-\frac{k}{n}\right)\left(W\left(t-\frac{k}{n}+\frac{1}{n}\right)-W\left(t-\frac{k}{n}\right)\right),$$

where $\mathbf{b} = \{b_k\}_{k=\kappa+1}^{\infty}$ is a sequence that must satisfy

$$b_k \in [k-1, k] \setminus \{0\}, \quad k \geqslant \kappa + 1,$$

but otherwise can be chosen freely.

Hybrid scheme

The hybrid scheme for $X(t)$ is then given by

$$X(t) \approx X_n(t) := \check{X}_n(t) + \hat{X}_n(t).$$

▸ Implementation

Remark

Define $\mathbf{b}_0 := \{k\}_{k=\kappa+1}^{\infty}$. Then in the case $\kappa = 0$ and $\mathbf{b} = \mathbf{b}_0$ we recover the approximation by Riemann sums.

Assumption

IV We have $N_n \sim n^{\gamma+1}$ as $n \to \infty$ for some $\gamma > 0$.

Asymptotics of the mean square error

Theorem

Suppose that $\gamma > -\frac{2\alpha+1}{2\beta+1}$ *and that for some* $\delta > 0$,

$$\mathbf{E}[|\sigma(s) - \sigma(0)|^2] = \mathcal{O}\big(s^{2\alpha+1+\delta}\big), \quad s \to 0+.$$

Then for all $t \in \mathbb{R}$,

$$\mathbf{E}[|X(t) - X_n(t)|^2] \sim J(\alpha, \kappa, \mathbf{b})\mathbf{E}[\sigma(0)^2] n^{-(2\alpha+1)} L_g(1/n)^2, \quad n \to \infty,$$

where

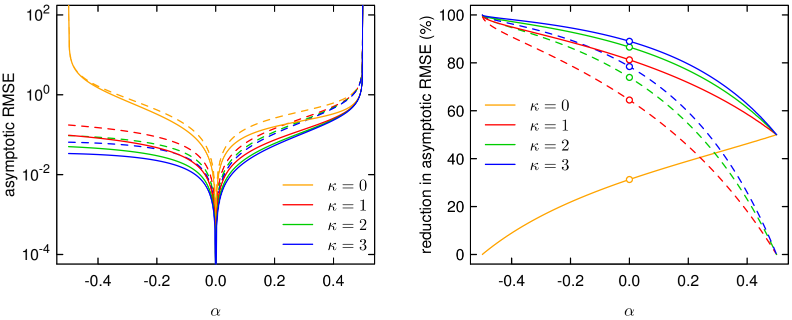
$$J(\alpha, \kappa, \mathbf{b}) := \sum_{k=\kappa+1}^{\infty} \int_{k-1}^{k} (y^\alpha - b_k^\alpha)^2 dy < \infty.$$

Asymptotic root mean square error

- The quantity $\sqrt{J(\alpha, \kappa, \mathbf{b})}$ can be seen as the asymptotic RMSE of the hybrid scheme.
- For any $\alpha \in \left(-\frac{1}{2}, \frac{1}{2}\right) \setminus \{0\}$, we can find **b** that minimizes $\sqrt{J(\alpha, \kappa, \mathbf{b})}$. We denote the minimizer by $\mathbf{b}^*$.
- It is illuminating to assess the asymptotic improvement on the approximation by Riemann sums:

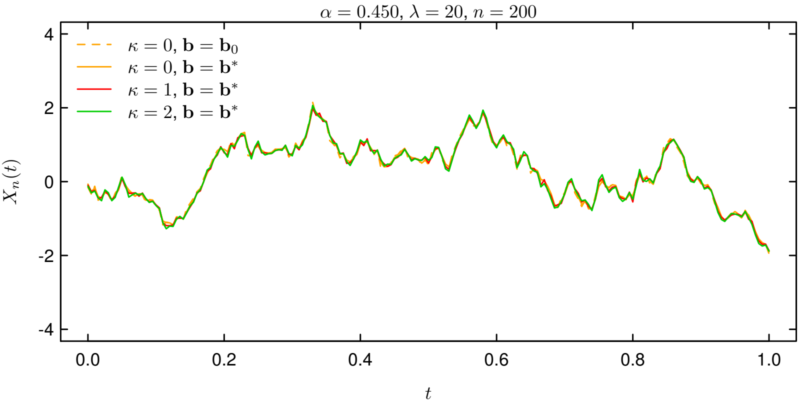
$$\text{reduction in asymptotic RMSE} = \frac{\sqrt{J(\alpha, \kappa, \mathbf{b})} - \sqrt{J(\alpha, 0, \mathbf{b}_0)}}{\sqrt{J(\alpha, 0, \mathbf{b}_0)}} \cdot 100\%.$$

Asymptotic root mean square error



Solid line: $\mathbf{b} = \mathbf{b}^*$; dashed line: $\mathbf{b} = \mathbf{b}_0$.

Simulated trajectories



Using $g(x) = c_{\alpha,\lambda} x^{\alpha} e^{-\lambda x}$, with $c_{\alpha,\lambda}$ such that $\int_0^{\infty} g(x)^2 \mathrm{d}x = 1$.

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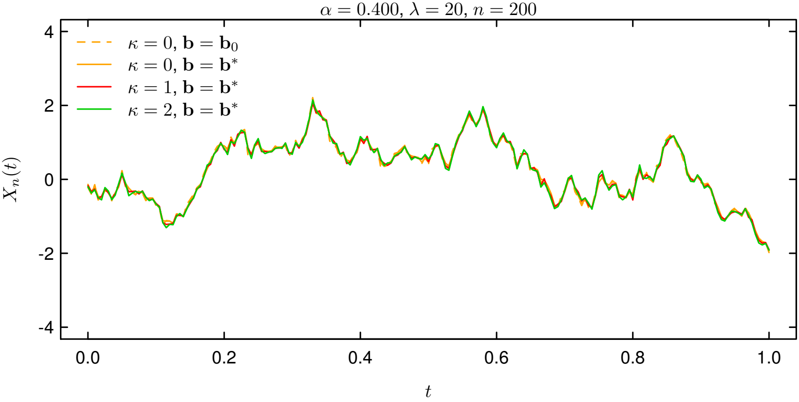

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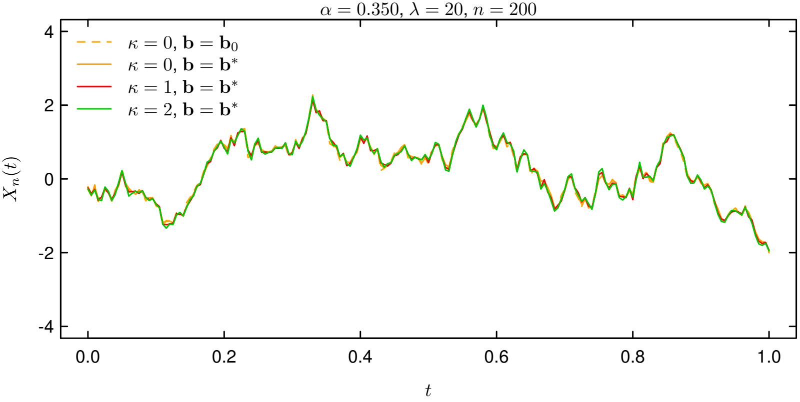

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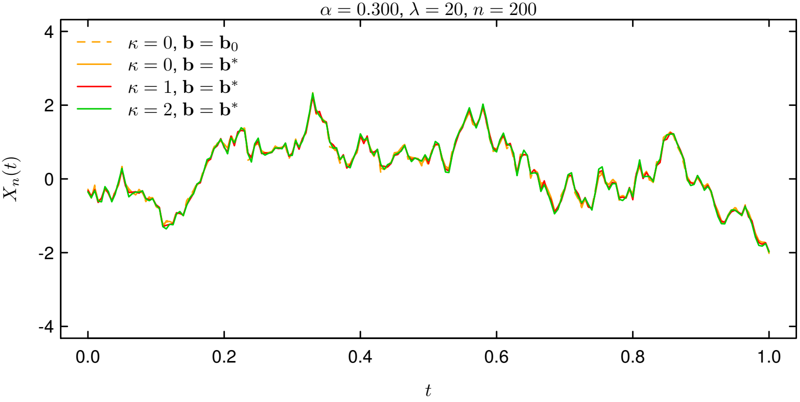

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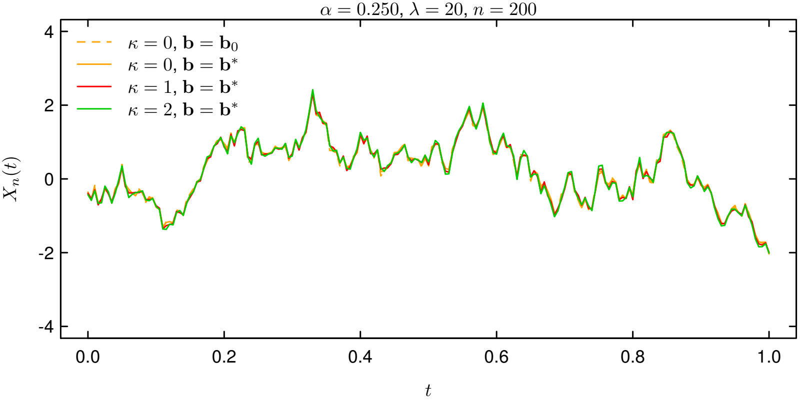

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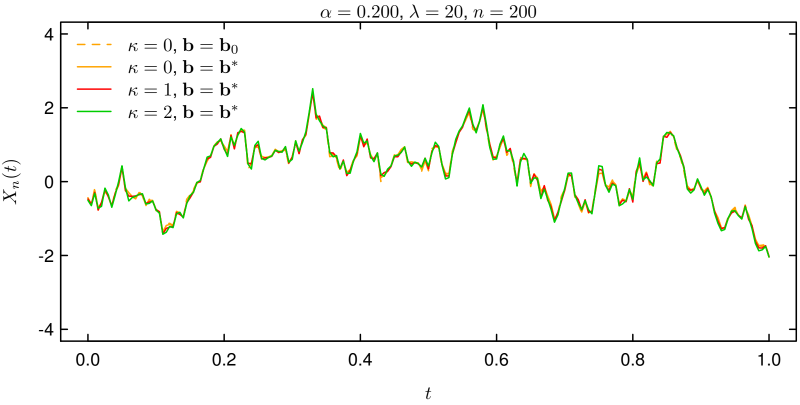

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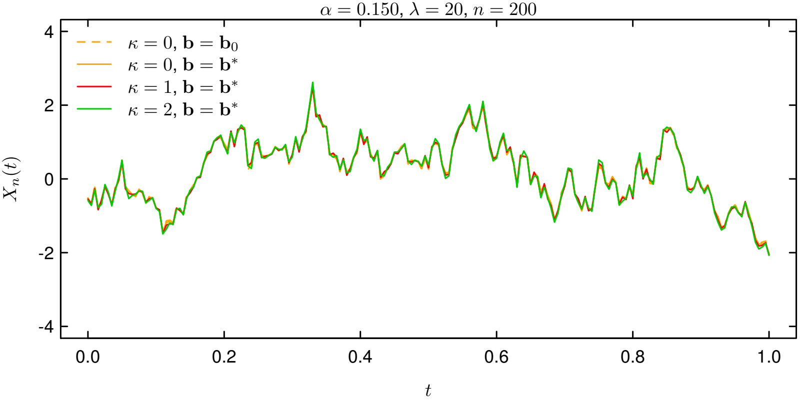

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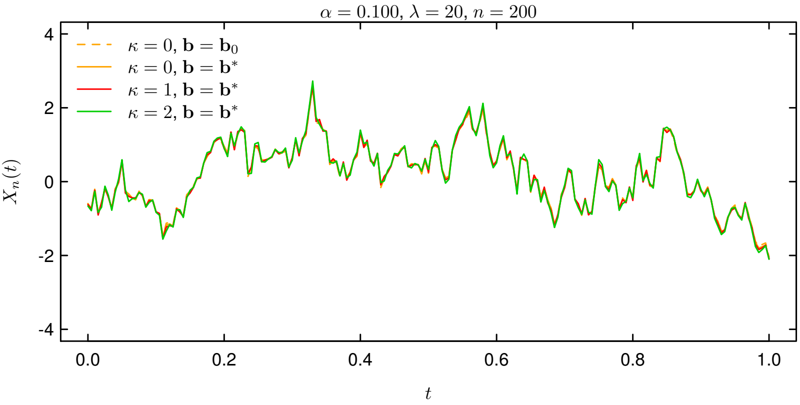

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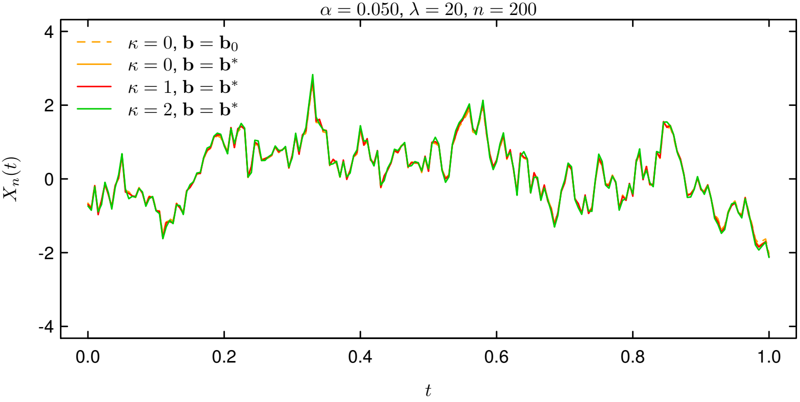

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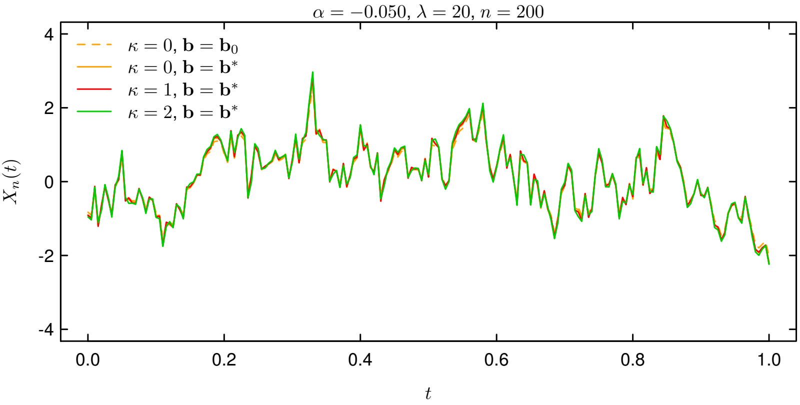

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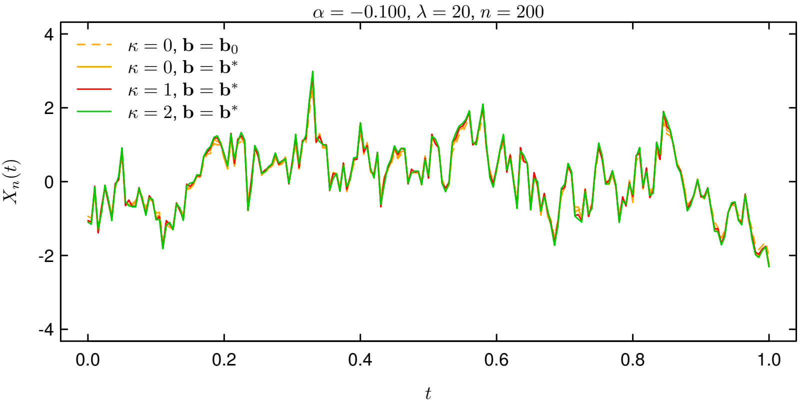

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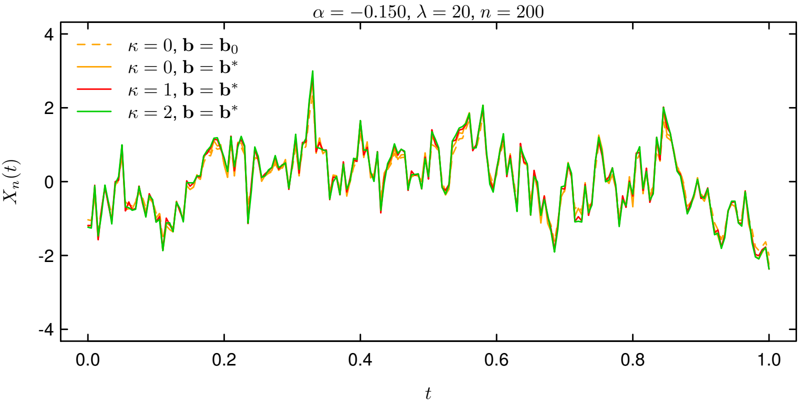

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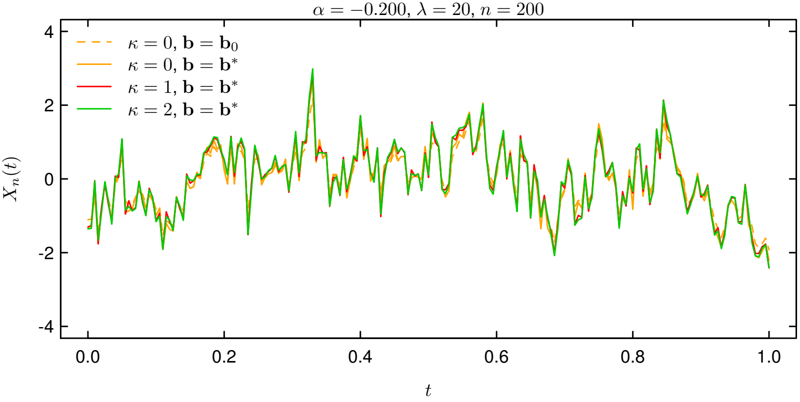

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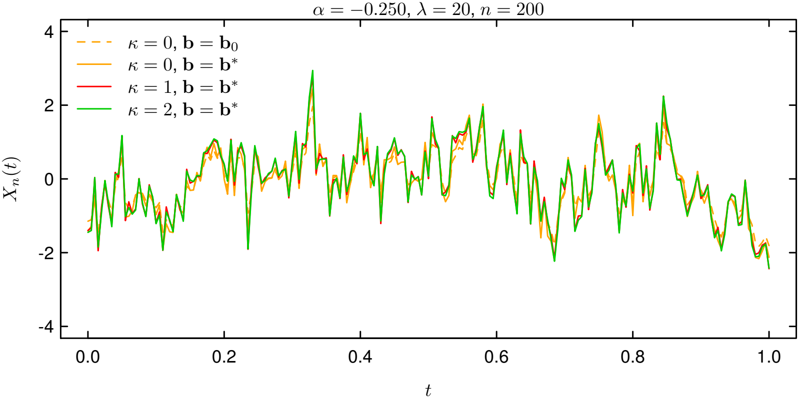

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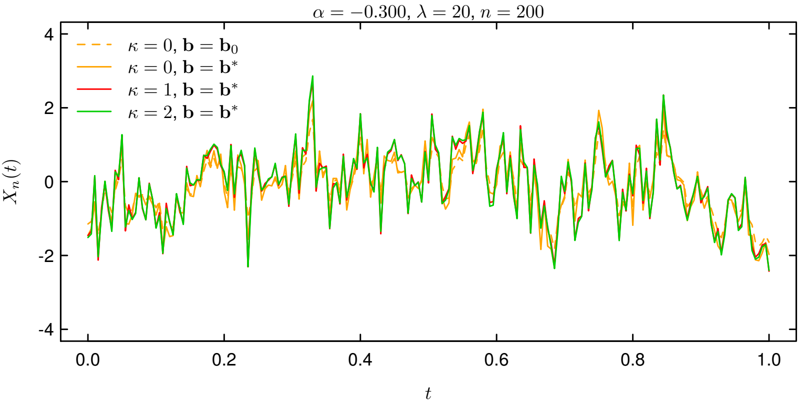

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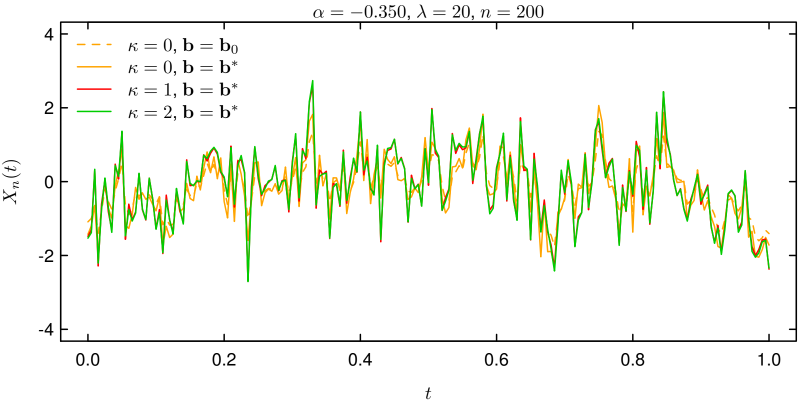

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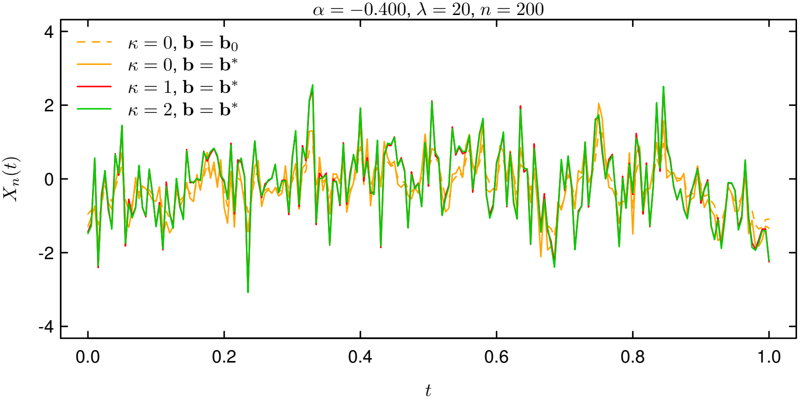

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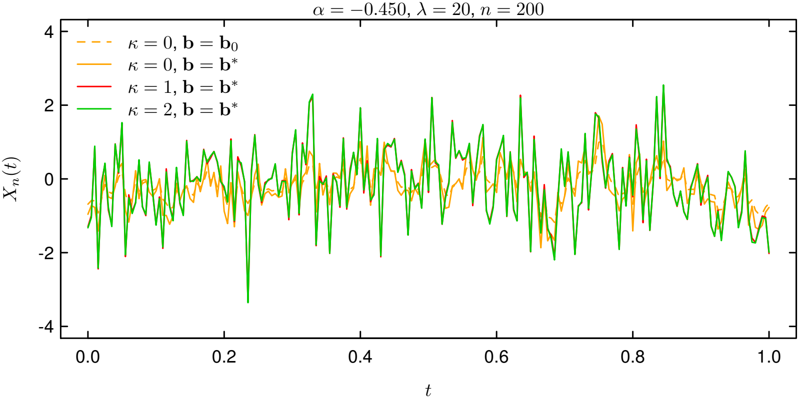

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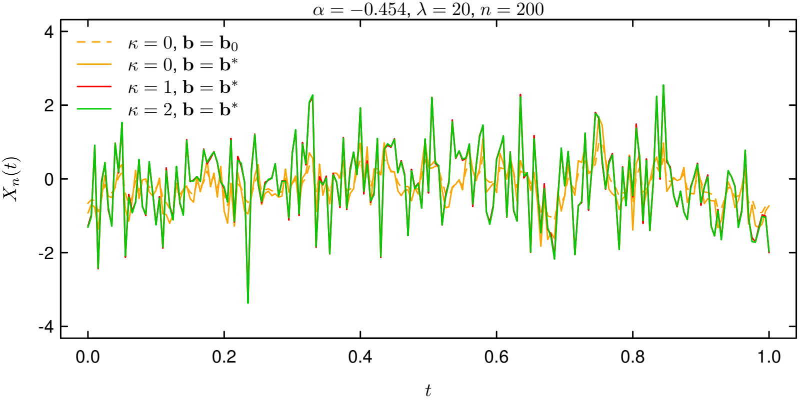

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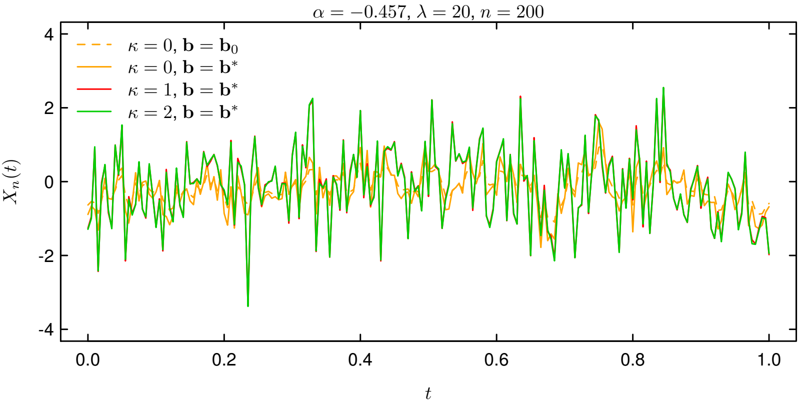

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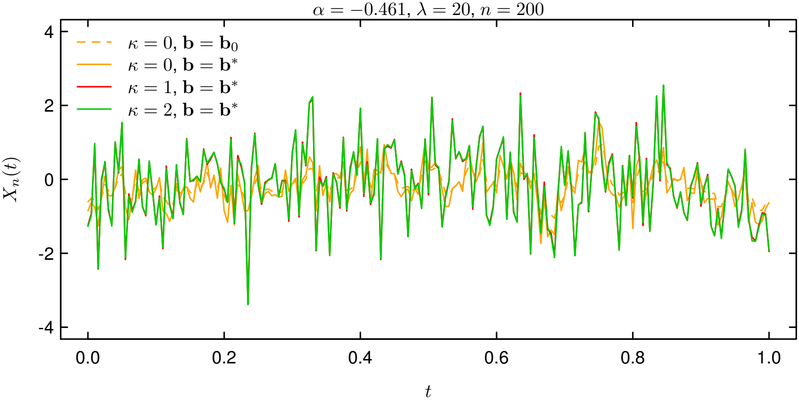

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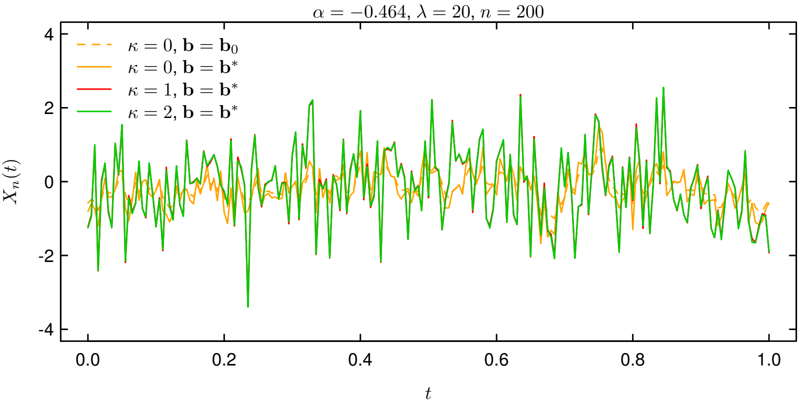

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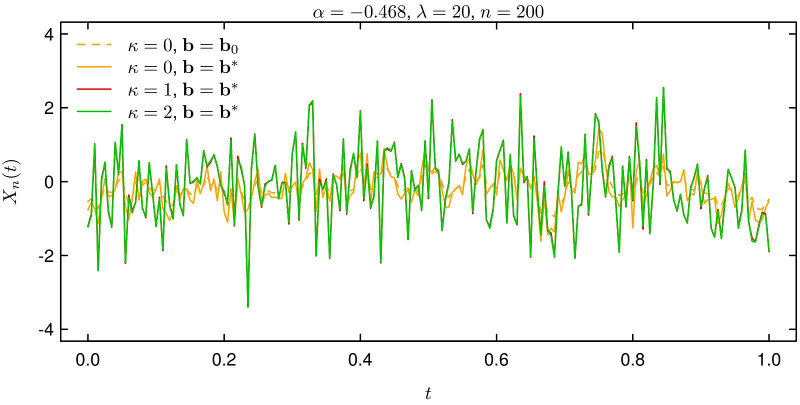

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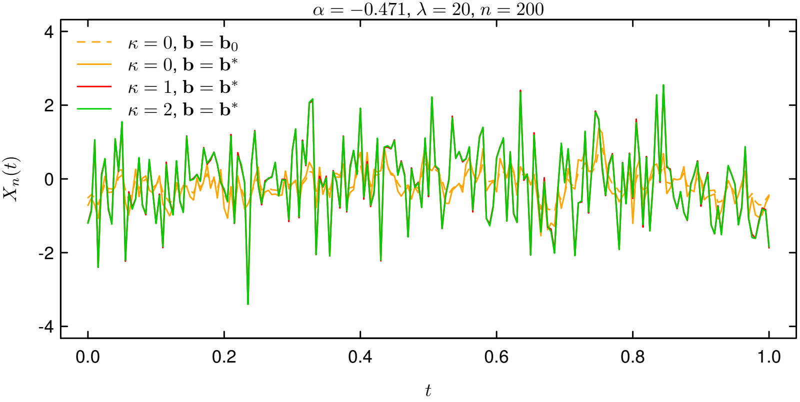

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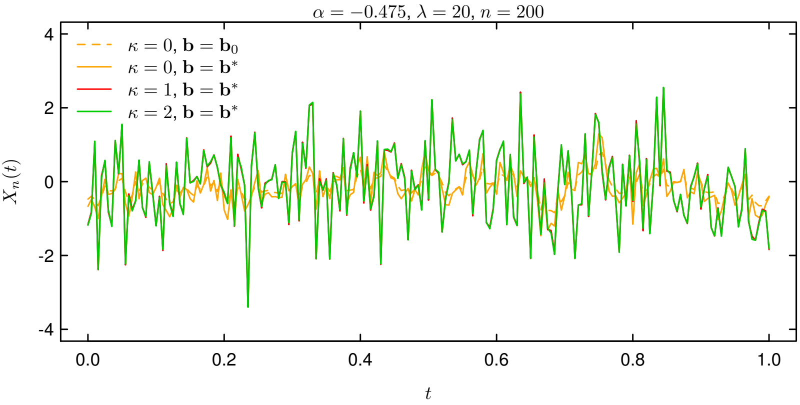

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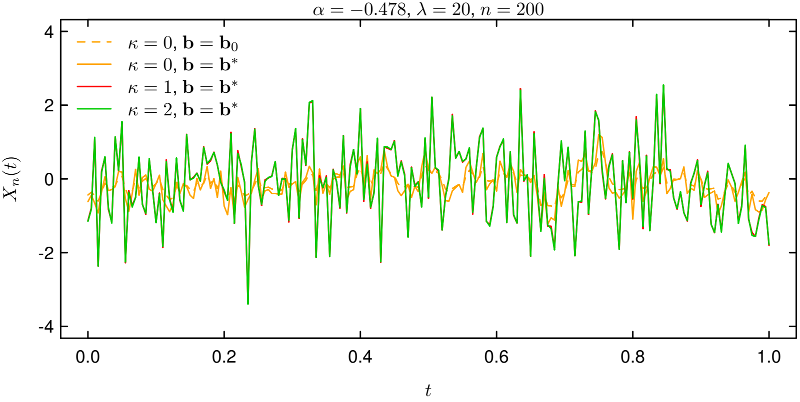

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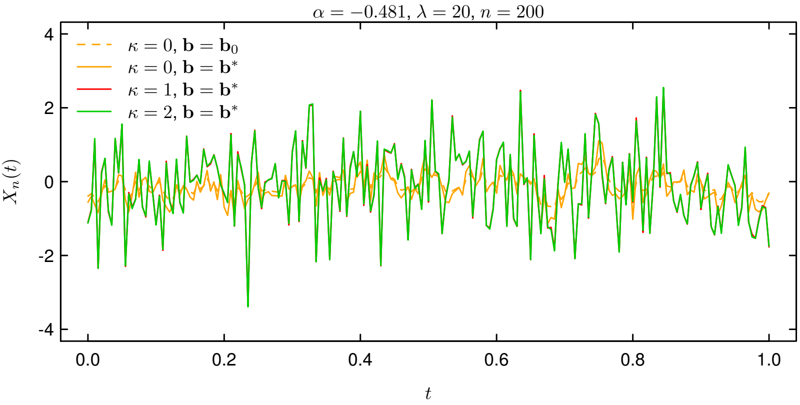

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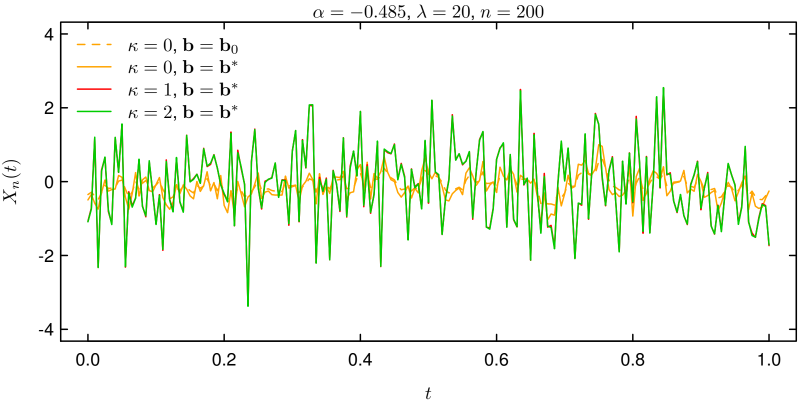

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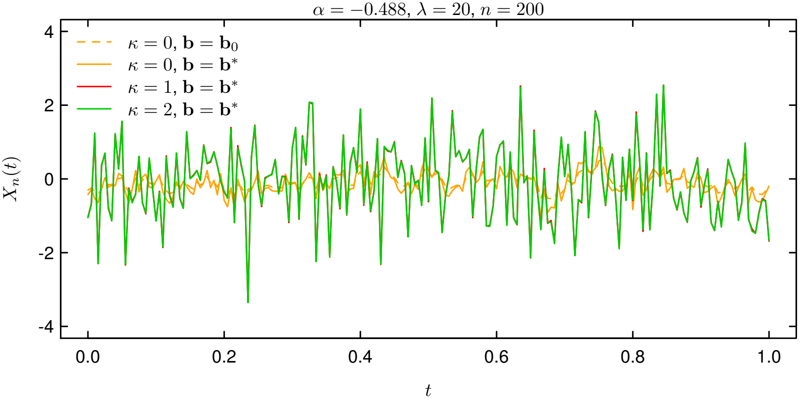

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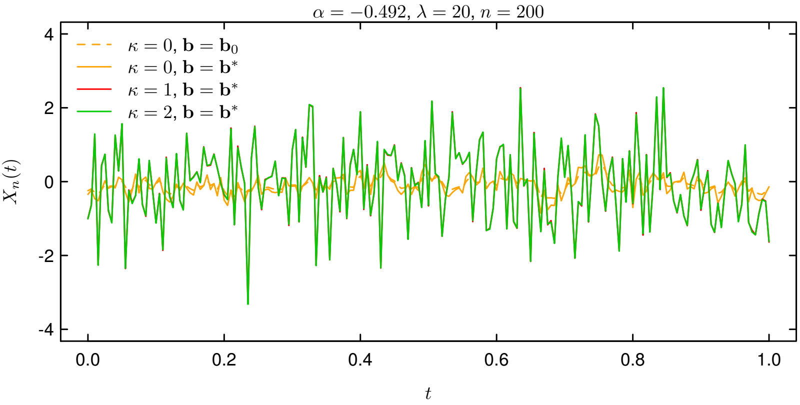

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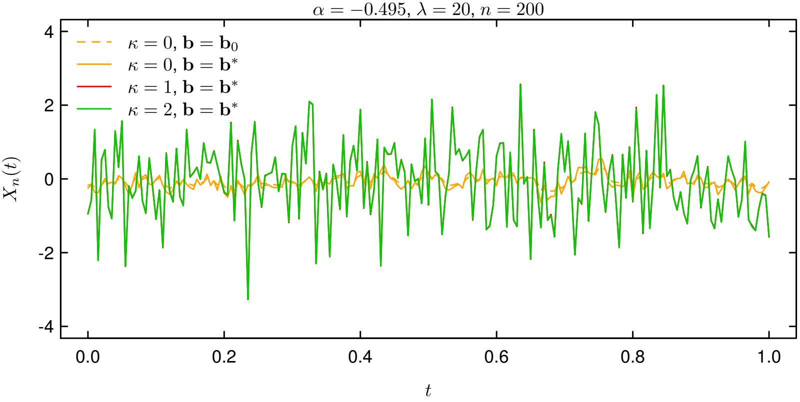

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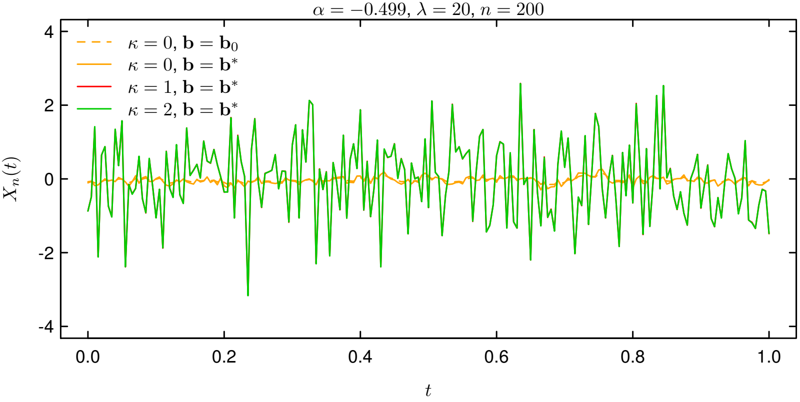

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Introduction

Hybrid scheme

Application to rough Bergomi model

Rough Bergomi model

The rough Bergomi model (Bayer et al., 2015) is an SV model, where the log of spot variance follows a rough Gaussian process.

More specifically, under an EMM, the stock price follows

$$S(t) := S(0) \exp\left(\int_0^t \sqrt{v(s)} \mathrm{d}B(s) - \frac{1}{2} \int_0^t v(s) \mathrm{d}s \right),$$

where

$$v(t) := \xi \exp\left(\underbrace{\eta \sqrt{2\alpha + 1} \int_0^t (t-s)^{\alpha} \mathrm{d}W(s)}_{=: Y(t)} - \frac{\eta^2}{2} t^{2\alpha+1} \right),$$

with $S_0, \xi, \eta > 0$, $\mathrm{d}\langle B, W \rangle_t = \rho \mathrm{d}t$, and $\alpha \in \left(-\frac{1}{2}, 0 \right)$.

Implied volatility smile

Bayer et al. (2015) have shown that the implied volatility smile $k \mapsto IV(k, T)$ corresponding to the call option price

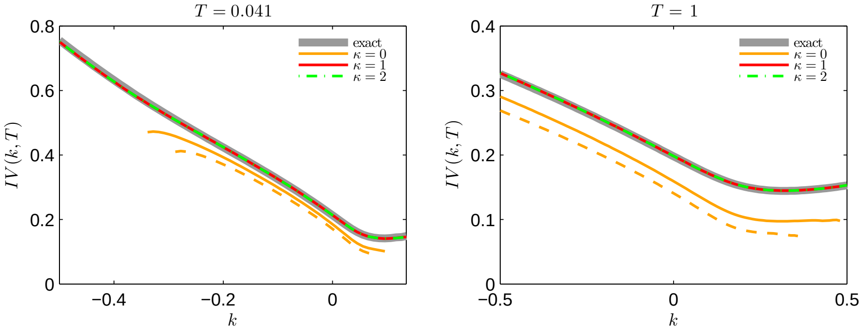
$$C(k, T) := \mathbf{E}[(S_T - S_0 e^k)^+]$$

fits nicely to empirical IV smiles when $\alpha \approx -0.4$.

To find the IV smile, Bayer et al. (2015) evaluate $C(k, T)$ numerically by Monte Carlo, simulating (B, Y) using an exact scheme.

- This simulation step can be made more efficient — without sacrificing accuracy — by using a (modified) hybrid scheme to simulate Y.

IV smile using the hybrid scheme



Solid/patterned line: $\mathbf{b} = \mathbf{b}^*$; dashed line: $\mathbf{b} = \mathbf{b}_0$.

$S(0)$	ξ	η	α	ρ
1	0.235^2	1.9	-0.43	-0.9

References

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Slow variation at zero

Definition

A function $L : (0, 1] \to [0, \infty)$ is slowly varying at 0 if for any $t > 0$,

$$\lim_{x \to 0+} \frac{L(tx)}{L(x)} = 1.$$

The intuition is that such a slowly varying function varies "less" than any power function "near" zero. Examples:

- If $\lim_{x \to 0+} L(x) \in (0, \infty)$ exists, then L is slowly varying.
- The function $L(x) = -\log x$ is slowly varying.

Back to assumptions

Implementation of the hybrid scheme

Outline of implementation

Generating $X_n\left(\frac{i}{n}\right)$ for $i = 0, 1, \ldots, \lfloor nT \rfloor$ involves:

1. sampling $\lfloor nT \rfloor + N_n$ IID observations from a $\kappa + 1$ dimensional Gaussian distribution,
2. generating a discretization of σ using some appropriate scheme,
3. computing the observations by summation and discrete convolution (using FFT).

- Glossing over the simulation of σ, the computational complexity of this procedure is $\mathcal{O}(N_n \log N_n) = \mathcal{O}(n^{1+\gamma} \log n)$.
- The computational complexity of an exact simulation in the Gaussian case would be $\mathcal{O}(n^3)$ (using Cholesky decomp.).

Back to hybrid scheme