The strong predictable representation property in initially enlarged filtrations

Claudio Fontana

Laboratoire de Probabilités et Modèles Aléatoires
Université Paris Diderot (Paris VII)

(http://sites.google.com/site/fontanaclaud)

London-Paris Bachelier Workshop on Mathematical Finance,
King's College London, September 25-26, 2015

Introduction

Transferring the martingale representation property onto an enlarged filtration

Problem description:

- Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a stochastic basis supporting a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$;
- let $S = (S_t)_{t \geq 0}$ be a local martingale on $(\Omega, \mathbb{F}, \mathbb{P})$;
- suppose that all $\mathbb{F}$-local martingales can be represented as stochastic integrals of S, i.e., S has the martingale representation property on $(\Omega, \mathbb{F}, \mathbb{P})$;

Introduction

Transferring the martingale representation property onto an enlarged filtration

Problem description:

- Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a stochastic basis supporting a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$;
- let $S = (S_t)_{t \geq 0}$ be a local martingale on $(\Omega, \mathbb{F}, \mathbb{P})$;
- suppose that all $\mathbb{F}$-local martingales can be represented as stochastic integrals of S, i.e., S has the martingale representation property on $(\Omega, \mathbb{F}, \mathbb{P})$;
- let $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ be a filtration on $(\Omega, \mathcal{A})$ such that $\mathcal{F}_t \subseteq \mathcal{G}_t$, for all $t \geq 0$;

Problem description:

- Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a stochastic basis supporting a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$;
- let $S = (S_t)_{t\geq 0}$ be a local martingale on $(\Omega, \mathbb{F}, \mathbb{P})$;
- suppose that all $\mathbb{F}$-local martingales can be represented as stochastic integrals of S, i.e., S has the martingale representation property on $(\Omega, \mathbb{F}, \mathbb{P})$;
- let $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ be a filtration on $(\Omega, \mathcal{A})$ such that $\mathcal{F}_t \subseteq \mathcal{G}_t$, for all $t \geq 0$;
- does the martingale representation property hold on $(\Omega, \mathbb{G}, \mathbb{P})$ with respect to some $\mathbb{G}$-local martingale?

Introduction

Problem description:

- Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a stochastic basis supporting a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$;
- let $S = (S_t)_{t\geq 0}$ be a local martingale on $(\Omega, \mathbb{F}, \mathbb{P})$;
- suppose that all $\mathbb{F}$-local martingales can be represented as stochastic integrals of S, i.e., S has the martingale representation property on $(\Omega, \mathbb{F}, \mathbb{P})$;
- let $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ be a filtration on $(\Omega, \mathcal{A})$ such that $\mathcal{F}_t \subseteq \mathcal{G}_t$, for all $t \geq 0$;
- does the martingale representation property hold on $(\Omega, \mathbb{G}, \mathbb{P})$ with respect to some $\mathbb{G}$-local martingale?

Why is it interesting to look at this problem?

- Enlarged filtrations naturally arise in financial mathematics (insider information, credit risk, insurance modeling...);
- martingale representation results have fundamental applications in the context of hedging, portfolio optimisation, BSDEs...

Setting

Initially enlarged filtrations and Jacod's density hypothesis

The basic ingredients:

1. a stochastic basis $(\Omega, \mathcal{A}, \mathbb{P})$ supporting a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$;
2. an $\mathbb{R}^d$-valued local martingale $S = (S_t)_{t \geq 0}$ on $(\Omega, \mathbb{F}, \mathbb{P})$;

Setting

Initially enlarged filtrations and Jacod's density hypothesis

The basic ingredients:

1. a stochastic basis $(\Omega, \mathcal{A}, \mathbb{P})$ supporting a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$;
2. an $\mathbb{R}^d$-valued local martingale $S = (S_t)_{t \geq 0}$ on $(\Omega, \mathbb{F}, \mathbb{P})$;
3. a random variable L taking values in a Lusin space $(E, \mathcal{B}_E)$, with unconditional law $\lambda : \mathcal{B}_E \to [0, 1]$;
4. the initially enlarged filtration $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ defined as the right-continuous augmentation of $\mathbb{G}^0 = (\mathcal{G}^0_t)_{t \geq 0}$, with $\mathcal{G}^0_t := \mathcal{F}_t \vee \sigma(L)$, $t \geq 0$.

Setting

Initially enlarged filtrations and Jacod's density hypothesis

The basic ingredients:

1. a stochastic basis $(\Omega, \mathcal{A}, \mathbb{P})$ supporting a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$;
2. an $\mathbb{R}^d$-valued local martingale $S = (S_t)_{t \geq 0}$ on $(\Omega, \mathbb{F}, \mathbb{P})$;
3. a random variable L taking values in a Lusin space $(E, \mathcal{B}_E)$, with unconditional law $\lambda : \mathcal{B}_E \to [0,1]$;
4. the initially enlarged filtration $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ defined as the right-continuous augmentation of $\mathbb{G}^0 = (\mathcal{G}^0_t)_{t \geq 0}$, with $\mathcal{G}^0_t := \mathcal{F}_t \vee \sigma(L)$, $t \geq 0$.

Let $\nu_t : \Omega \times \mathcal{B}_E \to [0,1]$ be a regular version of the $\mathcal{F}_t$-conditional law of L.

Key assumption (Jacod's density hypothesis)

For all $t \geq 0$, it holds that $\nu_t \ll \lambda$ $\mathbb{P}$-a.s.

Remarks:

- this assumption has been introduced in Jacod (1985) and it implies that every $\mathbb{F}$-semimartingale is also a $\mathbb{G}$-semimartingale (*H'-hypothesis*);
- we do not assume that $\nu_t \sim \lambda$.

Problem formulation

The strong predictable representation property from $\mathbb{F}$ to $\mathbb{G}$

Suppose that

$$\mathcal{M}_{\mathrm{loc}}(\mathbb{P},\mathbb{F}) = \Big\{\zeta + (\varphi \cdot S) : \zeta \in L^0(\mathcal{F}_0) \text{ and } \varphi \in L_m(S;\mathbb{P},\mathbb{F})\Big\},$$

i.e., S has the *strong predictable representation property (PRP)* on $(\Omega,\mathbb{F},\mathbb{P})$.

Does there exist a $\mathbb{G}$-local martingale $S^{\mathbb{G}}$ such that

$$\mathcal{M}_{\mathrm{loc}}(\mathbb{P},\mathbb{G}) = \Big\{\zeta + (\varphi \cdot S^{\mathbb{G}}) : \zeta \in L^0(\mathcal{G}_0) \text{ and } \varphi \in L_m(S^{\mathbb{G}};\mathbb{P},\mathbb{G})\Big\}?$$

If yes, how is $S^{\mathbb{G}}$ related to S?

Problem formulation

The strong predictable representation property from $\mathbb{F}$ to $\mathbb{G}$

Suppose that

$$\mathcal{M}_{\text{loc}}(\mathbb{P},\mathbb{F}) = \Big\{\zeta + (\varphi \cdot S) : \zeta \in L^0(\mathcal{F}_0) \text{ and } \varphi \in L_m(S;\mathbb{P},\mathbb{F})\Big\},$$

i.e., S has the *strong predictable representation property (PRP)* on $(\Omega,\mathbb{F},\mathbb{P})$.

Does there exist a $\mathbb{G}$-local martingale $S^{\mathbb{G}}$ such that

$$\mathcal{M}_{\text{loc}}(\mathbb{P},\mathbb{G}) = \Big\{\zeta + (\varphi \cdot S^{\mathbb{G}}) : \zeta \in L^0(\mathcal{G}_0) \text{ and } \varphi \in L_m(S^{\mathbb{G}};\mathbb{P},\mathbb{G})\Big\}?$$

If yes, how is $S^{\mathbb{G}}$ related to S?

A brief overview of the literature

Martingale representation results in initially enlarged filtrations

Results of the type "PRP($\mathbb{F}$) $\Rightarrow$ PRP($\mathbb{G}$)" have already been established:

- Grorud & Pontier (1998): S is Brownian motion and $\mathbb{F}$ its filtration;
- Amendinger (2000): S is an $\mathbb{R}^d$-valued locally square-integrable martingale;
- Amendinger, Becherer and Schweizer (2003) and Callegaro, Jeanblanc and Zargari (2013): S is a general $\mathbb{R}^d$-valued local martingale.

A brief overview of the literature

Martingale representation results in initially enlarged filtrations

Results of the type "PRP($\mathbb{F}$) $\Rightarrow$ PRP($\mathbb{G}$)" have already been established:

- Grorud & Pontier (1998): S is Brownian motion and $\mathbb{F}$ its filtration;
- Amendinger (2000): S is an $\mathbb{R}^d$-valued locally square-integrable martingale;
- Amendinger, Becherer and Schweizer (2003) and Callegaro, Jeanblanc and Zargari (2013): S is a general $\mathbb{R}^d$-valued local martingale.

The common assumption in all the above papers is of the form $\nu_t \sim \lambda$, $t \geq 0$.

Remarks:

- assuming $\nu_t \sim \lambda$ allows to rely on a very useful and simple technique: there exists $\mathbb{Q} \sim \mathbb{P}$ (*martingale preserving p.m.*) such that
  - $\mathcal{F}_t$ and $\sigma(L)$ are independent under $\mathbb{Q}$, for every $t \geq 0$,
  - $\mathbb{Q}|_{\mathcal{F}_t} = \mathbb{P}|_{\mathcal{F}_t}$, for every $t \geq 0$, and $\mathbb{Q}|_{\sigma(L)} = \mathbb{P}|_{\sigma(L)}$;
- however, this assumption excludes many interesting cases, especially in relation to the modeling of insider information leading to arbitrage.

Preliminary results

The family of conditional densities of L

From now on, we shall always assume that $\nu_t \ll \lambda$ $\mathbb{P}$-a.s., for all $t \geq 0$.

Lemma (Jacod, 1985; Amendinger, 2000)

There exists a $(\mathcal{B}_E \otimes \mathcal{O}(\mathbb{F}))$-measurable function $(x, \omega, t) \mapsto q_t^x(\omega) \in \mathbb{R}_+$ such that

- $\nu_t(dx) = q_t^x \, \lambda(dx)$ holds $\mathbb{P}$-a.s., for every $t \geq 0$;
- $q^x = (q_t^x)_{t \geq 0}$ is an $\mathbb{F}$-martingale, for every $x \in E$.

Preliminary results

The family of conditional densities of L

From now on, we shall always assume that $\nu_t \ll \lambda$ $\mathbb{P}$-a.s., for all $t \geq 0$.

Lemma (Jacod, 1985; Amendinger, 2000)

There exists a $(\mathcal{B}_E \otimes \mathcal{O}(\mathbb{F}))$-measurable function $(x, \omega, t) \mapsto q_t^x(\omega) \in \mathbb{R}_+$ such that

- $\nu_t(dx) = q_t^x \, \lambda(dx)$ holds $\mathbb{P}$-a.s., for every $t \geq 0$;
- $q^x = (q_t^x)_{t \geq 0}$ is an $\mathbb{F}$-martingale, for every $x \in E$.

For any $t \geq 0$ and $(\mathcal{B}_E \otimes \mathcal{F}_t)$-measurable function $(x, \omega, t) \mapsto f_t^x(\omega)$, it holds that

$$\mathbb{E}[f_t^L] = \mathbb{E}\left[\int_E f_t^x q_t^x \, \lambda(dx)\right] = \int_E \mathbb{E}\left[f_t^x q_t^x\right] \lambda(dx).$$

Preliminary results

The family of conditional densities of L

From now on, we shall always assume that $\nu_t \ll \lambda$ $\mathbb{P}$-a.s., for all $t \geq 0$.

Lemma (Jacod, 1985; Amendinger, 2000)

There exists a $(\mathcal{B}_E \otimes \mathcal{O}(\mathbb{F}))$-measurable function $(x, \omega, t) \mapsto q_t^x(\omega) \in \mathbb{R}_+$ such that

- $\nu_t(dx) = q_t^x \, \lambda(dx)$ holds $\mathbb{P}$-a.s., for every $t \geq 0$;
- $q^x = (q_t^x)_{t \geq 0}$ is an $\mathbb{F}$-martingale, for every $x \in E$.

For any $t \geq 0$ and $(\mathcal{B}_E \otimes \mathcal{F}_t)$-measurable function $(x, \omega, t) \mapsto f_t^x(\omega)$, it holds that

$$\mathbb{E}[f_t^L] = \mathbb{E}\left[\int_E f_t^x q_t^x \, \lambda(dx)\right] = \int_E \mathbb{E}\left[f_t^x q_t^x\right] \lambda(dx).$$

The decomposition formula of Jacod (1985):

$$\hat{S}^{\mathbb{G}} := S - \frac{1}{q_-^x} \cdot \langle S, q^x \rangle \Big|_{x=L} \text{ is a } \mathbb{G}\text{-local martingale.}$$

Preliminary results

Optional $\mathbb{G}$-decomposition of $\mathbb{F}$-local martingales

For every $x \in E$, define $\eta^x := \inf\{t \geq 0 : q_t^x = 0 \text{ and } q_{t-}^x > 0\}$.

Preliminary results

Optional $\mathbb{G}$-decomposition of $\mathbb{F}$-local martingales

For every $x \in E$, define $\eta^x := \inf\{t \geq 0 : q^x_t = 0 \text{ and } q^x_{t-} > 0\}$.

Proposition (Aksamit, Choulli and Jeanblanc, 2015)

Suppose that the space $L^1(\Omega, \mathcal{A}, \mathbb{P})$ is separable and let

$$S^{\mathbb{G}} := S - \frac{1}{q^L} \cdot [S, q^L] + \left(\Delta S_{\eta^x} \mathbf{1}_{[\![\eta^x,\infty[\![}\right)^{p,\mathbb{F}}\Big|_{x=L},$$

with $(\Delta S_{\eta^x} \mathbf{1}_{[\![\eta^x,\infty[\![})^{p,\mathbb{F}}$ denoting a $\mathcal{B}_E$-measurable version of the dual $\mathbb{F}$-predictable projection of $\Delta S_{\eta^x} \mathbf{1}_{[\![\eta^x,\infty[\![}$. Then $S^{\mathbb{G}} = (S^{\mathbb{G}}_t)_{t\geq 0}$ is a $\mathbb{G}$-local martingale.

Remarks:

- this gives an *optional decomposition* of S as a semimartingale in $\mathbb{G}$, as opposed to the classical decomposition formula of Jacod (1985);
- the last term vanishes if $\mathbb{P}(\eta^x < \infty) = 0$ for λ-a.e. $x \in E$;
- this last condition also ensures the absence of arbitrages of the first kind in $\mathbb{G}$ (see Acciaio, F. and Kardaras, 2015; Aksamit, Choulli and Jeanblanc, 2015).

The main result

Theorem

Suppose that the space $L^1(\Omega, \mathcal{A}, \mathbb{P})$ is separable and that $S = (S_t)_{t \geq 0}$ has the strong PRP on $(\Omega, \mathbb{F}, \mathbb{P})$. Then $S^{\mathbb{G}} = (S_t^{\mathbb{G}})_{t \geq 0}$ has the strong PRP on $(\Omega, \mathbb{G}, \mathbb{P})$.

Remarks:

- Under the assumptions of the above theorem, every $\mathbb{G}$-local martingale $M = (M_t)_{t \geq 0}$ can be written in the form

$$M_t = M_0 + \int_0^t \varphi_u \, dS_u^{\mathbb{G}}, \qquad \text{for all } t \geq 0, \ \mathbb{P}\text{-a.s.}$$

  for a suitable $\mathbb{G}$-predictable process $\varphi = (\varphi_t)_{t \geq 0} \in L_m(S^{\mathbb{G}}; \mathbb{P}, \mathbb{G})$;
- no assumptions on the family of conditional densities $\{q^x : x \in E\}$ (besides their existence).

Two corollaries

Martingale representations under additional hypotheses on the conditional densities

Corollary

Suppose that $\mathbb{P}(\eta^x < \infty) = 0$ for λ-a.e. $x \in E$ and that $S = (S_t)_{t\geq 0}$ has the strong PRP on $(\Omega, \mathbb{F}, \mathbb{P})$. Then the $\mathbb{R}^{d+1}$-valued process $(1/q_t^L, S_t/q_t^L)_{t\geq 0}$ has the strong PRP on $(\Omega, \mathbb{G}, \mathbb{P})$.

Remarks:

- in particular, $(1/q_t^L, S_t/q_t^L)_{t\geq 0}$ is a $\mathbb{G}$-local martingale;
- the $\mathbb{G}$-local martingale $(1/q_t^L)_{t\geq 0}$ is a natural deflator for S in $\mathbb{G}$;
- the strong PRP can be transferred from $\mathbb{F}$ onto $\mathbb{G}$ up to a suitable "change of numéraire", here represented by q^L.

Two corollaries

Martingale representations under additional hypotheses on the conditional densities

Corollary

Suppose that $\mathbb{P}(\eta^x < \infty) = 0$ for λ-a.e. $x \in E$ and that $S = (S_t)_{t \geq 0}$ has the strong PRP on $(\Omega, \mathbb{F}, \mathbb{P})$. Then the $\mathbb{R}^{d+1}$-valued process $(1/q_t^L, S_t/q_t^L)_{t \geq 0}$ has the strong PRP on $(\Omega, \mathbb{G}, \mathbb{P})$.

Remarks:

- in particular, $(1/q_t^L, S_t/q_t^L)_{t \geq 0}$ is a $\mathbb{G}$-local martingale;
- the $\mathbb{G}$-local martingale $(1/q_t^L)_{t \geq 0}$ is a natural deflator for S in $\mathbb{G}$;
- the strong PRP can be transferred from $\mathbb{F}$ onto $\mathbb{G}$ up to a suitable "change of numéraire", here represented by q^L.

Corollary (Amendinger, Becherer, Schweizer, 2003)

Suppose that $\nu_t \sim \lambda$ $\mathbb{P}$-a.s., for all $t \in [0, T]$, for some fixed $T < \infty$, and that $S = (S_t)_{t \in [0,T]}$ has the strong PRP on $(\Omega, \mathbb{F}, \mathbb{P})$. Then $S = (S_t)_{t \in [0,T]}$ has the strong PRP on $(\Omega, \mathbb{G}, \mathbb{Q})$, with $d\mathbb{Q} = (q_0^L/q_T^L)d\mathbb{P}$.

Remark: $\mathbb{Q}$ is the martingale preserving probability measure.

An outline of the proof of the main theorem

1. Every $\mathbb{G}$-martingale $M = (M_t)_{t \geq 0}$ can be written as

$$M_t = m_t^L,$$

where $(x, \omega, t) \mapsto m_t^x(\omega)$ is a measurable function such that $(m_t^x q_t^x)_{t \geq 0}$ is an $\mathbb{F}$-martingale, for all $x \in E$;

An outline of the proof of the main theorem

1. Every $\mathbb{G}$-martingale $M = (M_t)_{t \geq 0}$ can be written as
$$M_t = m_t^L,$$
where $(x, \omega, t) \mapsto m_t^x(\omega)$ is a measurable function such that $(m_t^x q_t^x)_{t \geq 0}$ is an $\mathbb{F}$-martingale, for all $x \in E$;
2. since S has the strong PRP on $(\Omega, \mathbb{F}, \mathbb{P})$, we can write, for each $x \in E$,
$$m_t^x q_t^x = m_0^x q_0^x + \int_0^t K_u^x \, dS_u \,, \tag{1}$$
$$q_t^x = q_0^x + \int_0^t H_u^x \, dS_u \,; \tag{2}$$

An outline of the proof of the main theorem

1. Every $\mathbb{G}$-martingale $M = (M_t)_{t \geq 0}$ can be written as
$$M_t = m_t^L,$$
where $(x, \omega, t) \mapsto m_t^x(\omega)$ is a measurable function such that $(m_t^x q_t^x)_{t \geq 0}$ is an $\mathbb{F}$-martingale, for all $x \in E$;
2. since S has the strong PRP on $(\Omega, \mathbb{F}, \mathbb{P})$, we can write, for each $x \in E$,
$$m_t^x q_t^x = m_0^x q_0^x + \int_0^t K_u^x \, dS_u \,, \qquad (1)$$
$$q_t^x = q_0^x + \int_0^t H_u^x \, dS_u \,; \qquad (2)$$
3. noting that $q_t^L > 0$ $\mathbb{P}$-a.s. for all $t \geq 0$, we can write
$$M_t = m_t^L = \frac{m_t^L q_t^L}{q_t^L} = \frac{1}{q_t^L} (m_t^x q_t^x)\big|_{x=L}$$
and applying integration by parts together with (1)-(2) will give a stochastic integral representation in $\mathbb{G}$.

An outline of the proof of the main theorem

1. Every $\mathbb{G}$-martingale $M = (M_t)_{t \geq 0}$ can be written as

$$M_t = m^L_t ,$$

where $(x, \omega, t) \mapsto m^x_t(\omega)$ is a measurable function such that $(m^x_t q^x_t)_{t \geq 0}$ is an $\mathbb{F}$-martingale, for all $x \in E$;

2. since S has the strong PRP on $(\Omega, \mathbb{F}, \mathbb{P})$, we can write, for each $x \in E$,

$$m^x_t q^x_t = m^x_0 q^x_0 + \int_0^t K^x_u \, dS_u \, , \tag{1}$$

$$q^x_t = q^x_0 + \int_0^t H^x_u \, dS_u \, ; \tag{2}$$

3. noting that $q^L_t > 0$ $\mathbb{P}$-a.s. for all $t \geq 0$, we can write

$$M_t = m^L_t = \frac{m^L_t q^L_t}{q^L_t} = \frac{1}{q^L_t} (m^x_t q^x_t)\big|_{x=L}$$

and applying integration by parts together with (1)-(2) will give a stochastic integral representation in $\mathbb{G}$.

4. **Warning:** measurability in x of the integrands K^x and H^x?
Do the stochastic integrals in (1)-(2) make sense in the filtration $\mathbb{G}$?

The proof

A representation result for a parameterized family of $\mathbb{F}$-martingales

Proposition

Suppose that $S=(S_t)_{t\geq 0}$ has the strong PRP on $(\Omega,\mathbb{F},\mathbb{P})$. Let $(x,\omega,t)\mapsto m^x_t(\omega)$ be a $(\mathcal{B}_E\otimes\mathcal{O}(\mathbb{F}))$-measurable function such that $(m^x_t q^x_t)_{t\geq 0}$ is an $\mathbb{F}$-martingale, for all $x\in E$. Then there exists a $(\mathcal{B}_E\otimes\mathcal{P}(\mathbb{F}))$-measurable function $(x,\omega,t)\mapsto K^x_t(\omega)\in\mathbb{R}^d$ satisfying $K^x\in L_m(S;\mathbb{P},\mathbb{F})$, for all $x\in E$, and such that

$$m^x_t q^x_t = m^x_0 q^x_0 + \int_0^t K^x_u\, dS_u, \qquad (\mathbb{P}\otimes\lambda)\text{-a.e. for all } t\geq 0.$$

Remark:

- two alternative proofs:
  - by extending to the general case a representation result by Esmaeeli & Imkeller (2015) obtained in a Brownian filtration,
  - via random measures and fine properties of PRP, similarly as in Jacod (1985);
- letting $m\equiv 1$, it holds that

$$q^x_t = q^x_0 + \int_0^t H^x_u\, dS_u, \qquad (\mathbb{P}\otimes\lambda)\text{-a.e. for all } t\geq 0,$$

for a suitable $(\mathcal{B}_E\otimes\mathcal{P}(\mathbb{F}))$-measurable function $(x,\omega,t)\mapsto H^x_t(\omega)\in\mathbb{R}^d$.

The proof

An auxiliary representation result for $\mathbb{G}$-martingales

Proposition

Suppose that $S = (S_t)_{t \geq 0}$ has the strong PRP on $(\Omega, \mathbb{F}, \mathbb{P})$. Let $M = (M_t)_{t \geq 0}$ be a $\mathbb{G}$-martingale. Then there exists an $\mathbb{R}^d$-valued $\mathbb{G}$-predictable S-integrable process $(K_t^L)_{t \geq 0}$ such that

$$M_t = \frac{1}{q_t^L}\left(q_0^L M_0 + \int_0^t K_u^L \, dS_u\right), \qquad \mathbb{P}\text{-a.s. for all } t \geq 0,$$

where the integral is understood as a $\mathbb{G}$-semimartingale stochastic integral.

Remark: the proof follows by combining the first two steps together with the fact that every $\mathbb{F}$-semimartingale is also a $\mathbb{G}$-semimartingale.

The proof

An auxiliary representation result for $\mathbb{G}$-martingales

Proposition

Suppose that $S = (S_t)_{t\geq 0}$ has the strong PRP on $(\Omega, \mathbb{F}, \mathbb{P})$. Let $M = (M_t)_{t\geq 0}$ be a $\mathbb{G}$-martingale. Then there exists an $\mathbb{R}^d$-valued $\mathbb{G}$-predictable S-integrable process $(K^L_t)_{t\geq 0}$ such that

$$M_t = \frac{1}{q^L_t}\left(q^L_0 M_0 + \int_0^t K^L_u \, dS_u\right), \qquad \mathbb{P}\text{-a.s. for all } t \geq 0,$$

where the integral is understood as a $\mathbb{G}$-semimartingale stochastic integral.

Remark: the proof follows by combining the first two steps together with the fact that every $\mathbb{F}$-semimartingale is also a $\mathbb{G}$-semimartingale.

Finishing the proof:

1. get an explicit representation for $1/q^L_t$;
2. apply integration by parts;
3. express all integrators in terms of the $\mathbb{G}$-local martingale $S^{\mathbb{G}}$.

A financial application

Hedging of contingent claims under insider information

- Suppose that $\mathcal{F}_0$ is trivial and let $T < \infty$ be a finite time horizon;
- let the $\mathbb{F}$-martingale $S = (S_t)_{t\in[0,T]}$ represent the discounted prices of d risky assets and suppose that S has the strong PRP on $(\Omega, \mathbb{F}, \mathbb{P})$;

A financial application

Hedging of contingent claims under insider information

- Suppose that $\mathcal{F}_0$ is trivial and let $T < \infty$ be a finite time horizon;
- let the $\mathbb{F}$-martingale $S = (S_t)_{t \in [0,T]}$ represent the discounted prices of d risky assets and suppose that S has the strong PRP on $(\Omega, \mathbb{F}, \mathbb{P})$;
- for every non-negative bounded $\mathcal{F}_T$-measurable r.v. ξ there exists an admissible $\mathbb{F}$-strategy H such that $\xi = v^{\mathbb{F}}(\xi) + \int_0^T H_u \, dS_u$ $\mathbb{P}$-a.s.

A financial application

Hedging of contingent claims under insider information

- Suppose that $\mathcal{F}_0$ is trivial and let $T < \infty$ be a finite time horizon;
- let the $\mathbb{F}$-martingale $S = (S_t)_{t \in [0,T]}$ represent the discounted prices of d risky assets and suppose that S has the strong PRP on $(\Omega, \mathbb{F}, \mathbb{P})$;
- for every non-negative bounded $\mathcal{F}_T$-measurable r.v. ξ there exists an admissible $\mathbb{F}$-strategy H such that $\xi = v^{\mathbb{F}}(\xi) + \int_0^T H_u \, dS_u$ $\mathbb{P}$-a.s.

Proposition

Suppose that $S = (S_t)_{t \in [0,T]}$ has the strong PRP on $(\Omega, \mathbb{F}, \mathbb{P})$ and let ξ be a bounded non-negative random variable. Then the following hold:

1. if ξ is $\mathcal{G}_T$-measurable, then there exists an admissible $\mathbb{G}$-strategy H such that $\xi = v^{\mathbb{G}}(\xi) + \int_0^T H_u \, dS_u$ $\mathbb{P}$-a.s., with $v^{\mathbb{G}}(\xi) = \mathbb{E}[\xi / q_T^L | L]$;
2. if ξ is $\mathcal{F}_T$-measurable, then it holds that $v^{\mathbb{G}}(\xi) \leq v^{\mathbb{F}}(\xi)$.

Remarks:

- better information always has a non-negative value;
- in the enlarged filtration $\mathbb{G}$, hedging is always possible (regardless of the possibility of arbitrage profits).

Thank you for your attention!