Portfolio Optimisation: Shadow Prices and Fractional Brownian Motion

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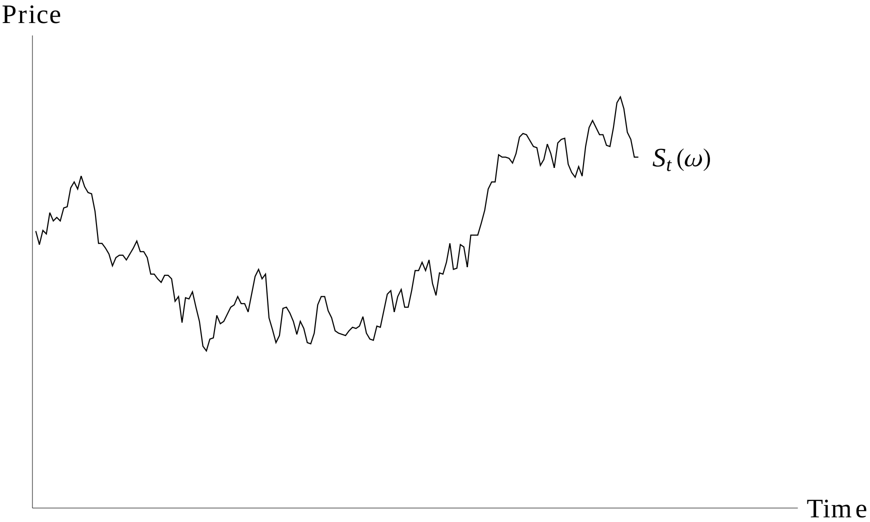
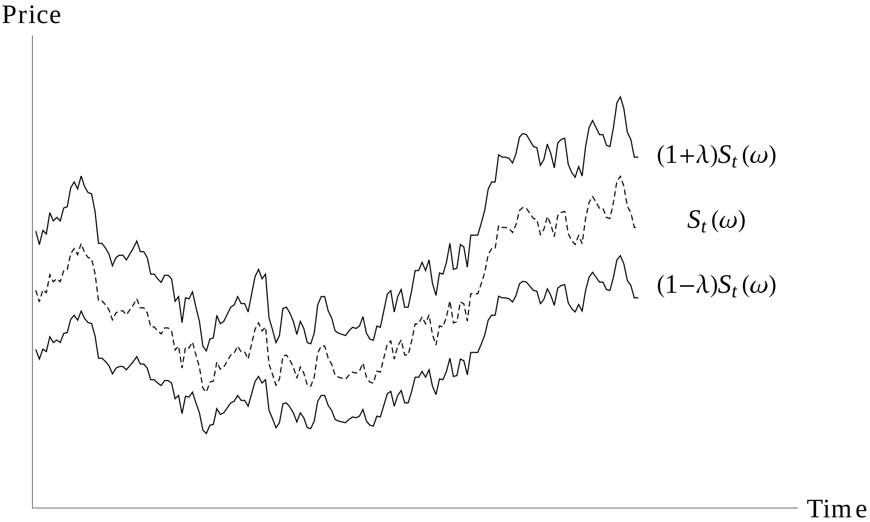
based on joint work(s) with Walter Schachermayer

Outline

Stock price



Bid-ask spread



Overview and comparison

	frictionless markets	markets with transaction costs
trading	buy and sell at **same** price S_t	buy at **higher** ask price $(1+\lambda)S_t$ sell at **lower** bid price $(1-\lambda)S_t$
"no arbitrage" price process	must be a semimartingale + very handy	can also be a **non-semimartingale** − more difficult to handle
critical Hölder exponent	either exactly 1 or exactly 2 − seems restrictive from a statistical point of view	any value in $[1,\infty)$ + more robust
optimal strategies	+ nice results for standard utilities	− hard to compute even for standard utilities and semimartingales
trading volume	− typically infinite, not possible in reality	+ automatically finite
summary	+ typically very handy − not always realistic	− more difficult to handle + more realistic

Financial markets with transaction costs

- Fix a strictly positive càdlàg **stochastic process** $S = (S_t)_{0 \le t \le T}$.
- A **self-financing trading strategy** under transaction costs $\lambda \in (0,1)$ is a predictable **finite variation** process $\varphi = (\varphi_t^0, \varphi_t^1)_{0 \le t \le T}$ such that

$$d\varphi_t^0 \le -(1+\lambda) S_t (d\varphi_t^1)^+ + (1-\lambda) S_t (d\varphi_t^1)^-.$$

- A self-financing strategy φ is **admissible**, if its **liquidation value**

$$\begin{aligned} V_t(\varphi) &:= \varphi_t^0 + (\varphi_t^1)^+ (1-\lambda) S_t - (\varphi_t^1)^- (1+\lambda) S_t \\ &= \varphi_0^0 + \varphi_0^1 S_0 + \int_0^t \varphi_s^1 dS_s - \lambda \int_0^t S_s d|\varphi^1|_s - \lambda S_t |\varphi_t^1| \\ &\ge -M \end{aligned}$$

for some $M > 0$ simultaneously for all $t \in [0, T]$.

- Denote by $\mathcal{A}^\lambda(x)$ the set of all self-financing and admissible trading strategies under transaction costs λ starting with $(\varphi_0^0, \varphi_0^1) = (x, 0)$.

Utility maximisation under transaction costs

- **Primal problem**: find **optimal trading strategy** $\widehat{\varphi} = (\widehat{\varphi}^0, \widehat{\varphi}^1)$ to

$$\text{maximise} \quad E\left[U\left(V_T(\varphi)\right)\right] := E\left[U\left(x + \int_0^T \varphi^1_u dS_u - \lambda \int_0^T S_u d|\varphi^1|_u\right)\right].$$

- **Dual problem**: find **optimal λ-consistent price system** $(\widehat{Z}^0, \widehat{Z}^1)$, i.e. local martingales $(Z^0, Z^1) > 0$ such that $\widetilde{S} := \frac{Z^1}{Z^0} \in [(1-\lambda)S, (1+\lambda)S]$, to

$$\text{minimise} \quad E\left[U^*\left(Z^0_T\right) + xZ^0_T\right].$$

- **Lagrange duality**: If $(\widehat{Z}^0, \widehat{Z}^1)$ exists, then $V_T(\widehat{\varphi}) = (U')^{-1}(\widehat{Z}^0_T)$.
- **Technical point**: Solution $(\widehat{Z}^0, \widehat{Z}^1)$ to dual problem is, in general, only a **limit** of consistent price systems, i.e., an **optional strong supermartingale**.

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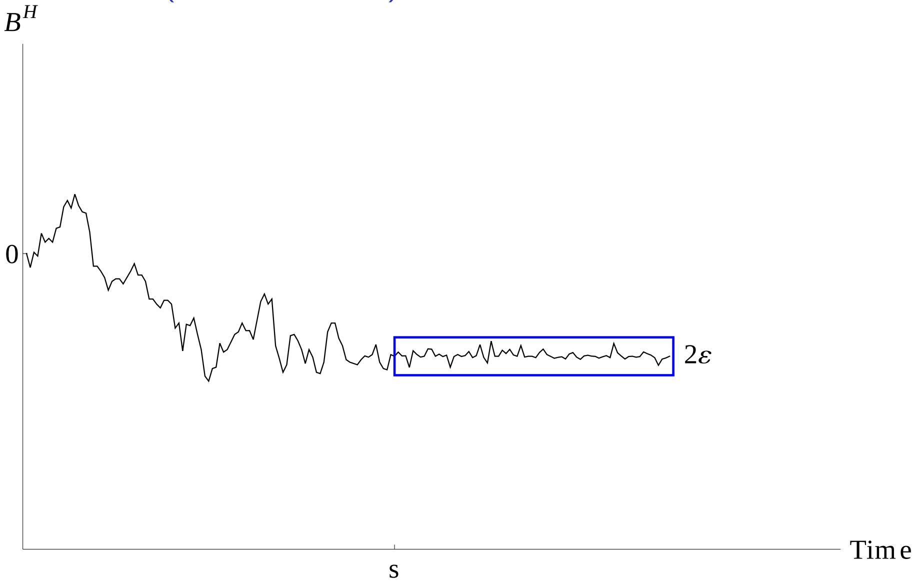
- In principle, the above allows also to consider **non-semimartingales** for S.
- So what about concrete examples?

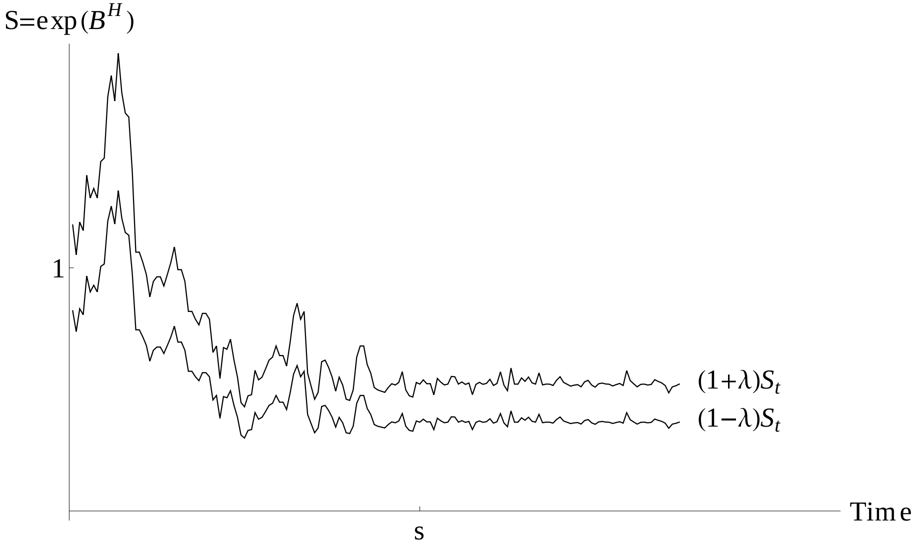
Fractional Brownian motion

- A "nice" class of **Gaussian processes** $B^H = (B^H_t)$ indexed by $H \in (0,1)$.
- Mandelbrot: Natural model for stock prices.
- Critical Hölder exponent is $\frac{1}{H}$ and can therefore take any value in $(1,\infty)$.
- Prime example of non-semimartingales for $H \neq \frac{1}{2}$.
- For frictionless trading, fractional models like the **fractional Black-Scholes model** $S = \exp(B^H)$ admit "arbitrage"; see e.g. Rogers (1997), Cheridito (2003) for explicit constructions.
- Guasoni (2006): The fractional Black-Scholes model is arbitrage-free under transaction costs, as fractional Brownian motion $B^H = \log(S)$ is **sticky**.

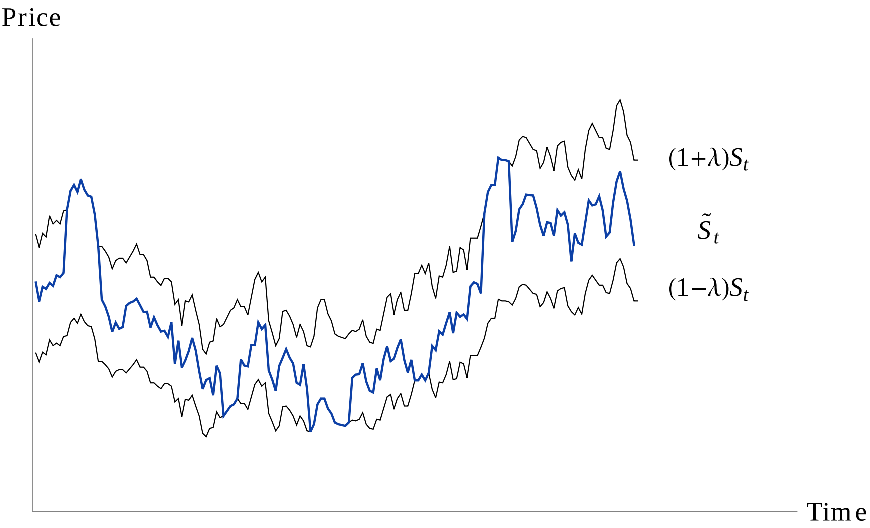
Stickiness (Guasoni 2006)



No arbitrage under transaction costs



Shadow price (Jouini/Kallal, Cvitanić/Karatzas)



Shadow price

Definition

A semimartingale price process $\widehat{S} = (\widehat{S}_t)$ *is a* **shadow price**, *if*

i) $\widehat{S}$ *is valued in the bid-ask spread* $[(1-\lambda)S, (1+\lambda)S]$.

ii) *The solution* $\widehat{\psi}$ *to the* **frictionless** *utility maximisation problem: to*

$$\text{maximise} \qquad E\big[U\big(V_T(\psi)\big)\big] := E\left[U\left(x + \int_0^T \psi_s d\widehat{S}_s\right)\right]$$

exists.

iii) $\widehat{\psi}$ *is of finite variation and "admissible" under transaction costs.*

iv) $\{d\widehat{\psi}^1 > 0\} \subseteq \{\widehat{S} = (1+\lambda)S\}$ *and* $\{d\widehat{\psi}^1 < 0\} \subseteq \{\widehat{S} = (1-\lambda)S\}$.

Then $\widehat{\psi}$ *coincides with the solution* $\widehat{\varphi}$ *under transaction costs.*

Existence of shadow prices?

- Cvitanić/Karatzas (1996): Existence in an Itô process setting, **if** the solution to the dual problem is a local martingale. — Not clear under which conditions this is the case.
  - Kallsen/Muhle-Karbe (2011): finite probability space.
  - Explicit constructions for various concrete problems in the classical(!) Black-Scholes model; Kallsen/Muhle-Karbe (2009),...
  - Beyond the classical Black-Scholes model?
  - C./Deutsch/Forde/Zhang: Construction for geometric Ornstein-Uhlenbeck process.
  - No-shortselling (somewhat different problem); Loewenstein (2001), Benedetti/Campi/Kallsen/Muhle-Karbe (2011).
- No general results that apply to Cvitanić/Karatzas (1996) so far.
- Counter-examples in discrete time:
  - Benedetti/Campi/Kallsen/Muhle-Karbe (2011).
  - C./Muhle-Karbe/Schachermayer (2012).

Outline

Sufficient conditions

Theorem (C./Schachermayer/Yang)

Suppose that

0) $U : (0,\infty) \to \mathbb{R}$ *satisfies* $\limsup_{x\to\infty} \frac{xU'(x)}{U(x)} < 1$.

i) S *is continuous.*

ii) S *satisfies* **(NUPBR)** *or, equivalently, admits an ELMD.*

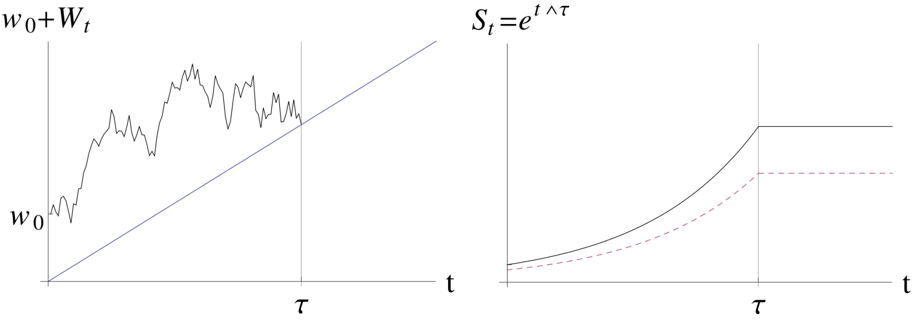
iii) $u(x) := \sup_{\varphi \in \mathcal{A}^\lambda(x)} E\big[U\big(V_T(\varphi)\big)\big] < \infty$.

Then $(\widehat{Z}^0, \widehat{Z}^1)$ *is a local martingale and* $\widehat{S} := \frac{\widehat{Z}^1}{\widehat{Z}^0}$ *a shadow price process.*

- Conditions can be verified without knowing the solution to the dual problem before; compare Cvitanić/Karatzas (1996).
- **Quite sharp:** There exist **counter-examples**, if i) or ii) are not satisfied.
- Condition ii), which implies that S is a **semimartingale**, **cannot** be replaced by the weaker condition that "S is sticky" typically used for fBm.

Example: S is continuous and sticky

- C./Schachermayer/Yang.



- S admits an **unbounded increasing profit** and hence **no** ELMM.

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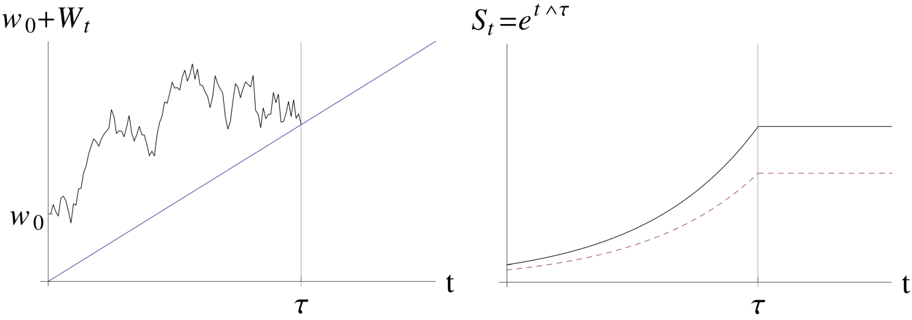

- S admits an **unbounded increasing profit** and hence **no** ELMM.
- **No** solution to **any** frictionless utility maximisation problem.

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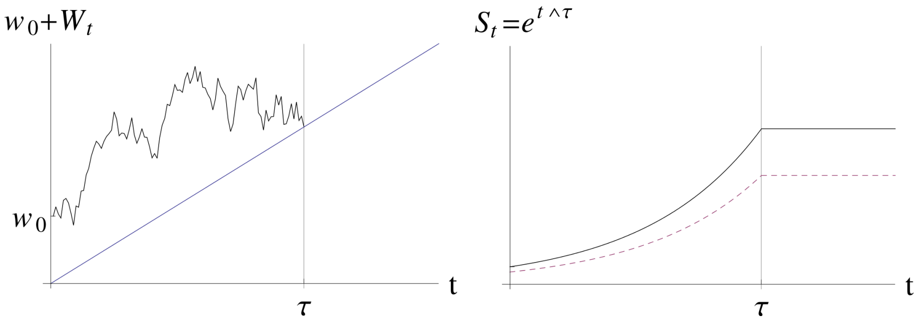

- S admits an **unbounded increasing profit** and hence **no** ELMM.
- **No** solution to **any** frictionless utility maximisation problem.
- However, S is **sticky** and S is arbitrage-free under transaction costs.

Example (cont.)

Proposition (C./Schachermayer/Yang)

There exists a **non-decreasing function** $\ell : [0,\infty) \to [0, \frac{1}{\lambda}]$ *such that the optimal strategy* $\widehat{\varphi} = (\widehat{\varphi}^0, \widehat{\varphi}^1)$ *to*

$$E\big[\log\big(V_\tau(\varphi)\big)\big] \to \max!, \qquad \varphi \in \mathcal{A}^\lambda(1),$$

is given by the **smallest non-decreasing process** $\widehat{\varphi}^1$ *such that*

i) $d\widehat{\varphi}^0_t = -(1+\lambda)S_t d\widehat{\varphi}^1_t$ *for all* $t \geq 0$.

ii) $\frac{1}{\lambda} \geq \frac{\widehat{\varphi}^1_t S_t}{\widehat{\varphi}^0_t + \widehat{\varphi}^1_t S_t} \geq \ell(w_0 + W_t - t)$ *for all* $t \geq 0$.

Moreover, there exists $\overline{w} \in (0,\infty)$ *such that* $\ell(w) = \frac{1}{\lambda}$ *for all* $w \geq \overline{w}$.

For $w_0 > \overline{w}$, we would therefore have

$$\widehat{S}_t = (1+\lambda)S_t \qquad \text{for all } t \leq \sigma := \inf\{s > 0 \mid w_0 + W_s - s < \overline{w}\}$$

for any candidate shadow price and hence **no shadow price** exists.

Sufficient conditions (cont.)

Theorem (C./Schachermayer/Yang)

Suppose that

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ii) S *satisfies* **no simple arbitrage (NSA)**.

iii) $u(x) := \sup_{\varphi \in \mathcal{A}^\lambda(x)} E\big[U\big(V_T(\varphi)\big)\big] < \infty$.

Then $(\widehat{Z}^0, \widehat{Z}^1)$ *is a local martingale and* $\widehat{S} := \frac{\widehat{Z}^1}{\widehat{Z}^0}$ *a shadow price process.*

- Is (NSA) satisfied for the **fractional Black-Scholes model**?

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- Condition iii) is satisfied for $U(x) = \frac{x^p}{p}$ with $p \in (-\infty, 0)$.

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Theorem (C./Schachermayer)

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iii) $E[U^*(\bar{Z}^0_T)] < +\infty$.

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- In principle, the above allows to consider **non-semimartingales** for S.
- For the **fractional Black-Scholes model** conditions i) and ii) are satisfied.
- So what about iii) for $U(x) = 1 - e^{-x}$? — Hard to verify directly.

Sufficient conditions (cont.)

Theorem (C./Schachermayer)

Suppose that

0) *$U : \mathbb{R} \to \mathbb{R}$ is bounded from above and satisfies $\liminf_{x \to -\infty} \frac{xU'(x)}{U(x)} > 1$.*

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- Conditions i)–ii) are satisfied for the **fractional Black-Scholes model**.
- Proof combines arguments from convex duality with the stickiness condition.
- By the change of measure $\frac{dP_B}{dP} = \frac{\exp(B)}{E[\exp(B)]}$ the above also gives the existence of **exponential utility indifference prices** for any claim $B \in L^\infty(P)$.

Outline

Future research

- **Future goals**:
  - **Quantitative** results for fractional models.
  - Understand impact of **non-semimartingality** on optimal strategy.
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  1) an **Itô process**, i.e.

$$d\widehat{S}_t = \widehat{S}_t \left(\widehat{\mu}_t dt + \widehat{\sigma}_t dW_t \right),$$

  2) evolving in the bid-ask spread $\widehat{S} \in [(1-\lambda)S, (1+\lambda)S]$ such that

  3) the optimal strategies coincide, i.e. $\widehat{\psi} = \widehat{\varphi}$, and

  4) $\{d\widehat{\varphi}^1 > 0\} \subseteq \{\widehat{S} = (1+\lambda)S\}$ and $\{d\widehat{\varphi}^1 < 0\} \subseteq \{\widehat{S} = (1-\lambda)S\}$.

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- **Basic idea:** Combine 1)–4) with results for utility maximisation for **Itô processes** to describe optimal strategy $\widehat{\varphi} = (\widehat{\varphi}^0, \widehat{\varphi}^1)$ more explicitly.

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- For the **fractional Black-Scholes model** $S = \exp(B^H)$ the shadow price is
  1) an **Itô process**, i.e.

$$d\widehat{S}_t = \widehat{S}_t \left(\widehat{\mu}_t dt + \widehat{\sigma}_t dW_t \right),$$

  2) evolving in the bid-ask spread $\widehat{S} \in [(1-\lambda)S, (1+\lambda)S]$ such that
  3) the optimal strategies coincide, i.e. $\widehat{\psi} = \widehat{\varphi}$, and
  4) $\{d\widehat{\varphi}^1 > 0\} \subseteq \{\widehat{S} = (1+\lambda)S\}$ and $\{d\widehat{\varphi}^1 < 0\} \subseteq \{\widehat{S} = (1-\lambda)S\}$.
- **Basic idea:** Combine 1)–4) with results for utility maximisation for **Itô processes** to describe optimal strategy $\widehat{\varphi} = (\widehat{\varphi}^0, \widehat{\varphi}^1)$ more explicitly.
- This then also gives results for **exponential utility indifference pricing** by comparing **two** shadow prices given by the Itô processes $\widehat{S}^B$ and $\widehat{S}$.

Future research

- **Future goals**:
  - **Quantitative** results for fractional models.
  - Understand impact of **non-semimartingality** on optimal strategy.
  - Utility-based **pricing** and **hedging** for fractional models.
- For the **fractional Black-Scholes model** $S = \exp(B^H)$ the shadow price is

  1) an **Itô process**, i.e.

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- **Basic idea:** Combine 1)–4) with results for utility maximisation for **Itô processes** to describe optimal strategy $\widehat{\varphi} = (\widehat{\varphi}^0, \widehat{\varphi}^1)$ more explicitly.
- This then also gives results for **exponential utility indifference pricing** by comparing **two** shadow prices given by the Itô processes $\widehat{S}^B$ and $\widehat{S}$.
- Importance: Superreplication price is too high by face-lifting theorems.

Summary

Sufficient conditions for existence of shadow prices:

1) S is continuous and satisfies (NSA) $U : (0, \infty) \to \mathbb{R}$. **Quite sharp.**

2) S is locally bounded and admits a $CPS^{\lambda'}$ $(\bar{Z}^0, \bar{Z}^1)$ for $\lambda' \in [0, \lambda)$ satisfying $E[U^*(\bar{Z}^0_T)] < \infty$ for $U : \mathbb{R} \to \mathbb{R}$.

3) S is continuous and sticky for $U : \mathbb{R} \to \mathbb{R}$ bounded from above.

Counter-examples for $U : (0, \infty) \to \mathbb{R}$:

- S is continuous and sticky are **not** sufficient.

Fractional Brownian motion:

- Existence of shadow price for bounded power and exponential utility.
- Shadow price is Itô process.
- Exploit connection to frictionless markets to obtain quantitative results.

Thank you for your attention and

for coming here on Saturday morning!

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Talk based on

C. Czichowsky and W. Schachermayer.
Strong supermartingales and limits of non-negative martingales.
Preprint, 2013. To appear in *The Annals of Probability*.

C. Czichowsky and W. Schachermayer.
Duality theory for portfolio optimisation under transaction costs.
Preprint, 2014. To appear in *The Annals of Applied Probability*.

C. Czichowsky, W. Schachermayer, and J. Yang.
Shadow prices for continuous price processes.
Preprint, 2014. To appear in *Mathematical Finance*.

C. Czichowsky and W. Schachermayer.
Portfolio optimisation beyond semimartingales: shadow prices and fractional Brownian motion.
Preprint, 2015.