Integrated structural approach to Counterparty Credit Risk with dependent jumps

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September 2015

Introduction

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- General context of the problem
  - Firm 1 (S_1) - Counterparty; Firm 2 (S_2) - Investor
  - **Counterparty Credit Risk**: risk of a party to a financial contract defaulting prior to/at the contract's expiration
  - **Credit Value Adjustment** (CVA):

    $$CVA_1 = (1 - R_1)\mathbb{E}\left(1_{(\tau_1 \leq \min(\tau_2, T))}\Psi^+(\tau_1; S_3, T)\right)$$

  - $\Psi(\cdot)$ - disc. value of OTC contract on S_3 (underlying asset)
  - $\tau_j = \inf\{t \geq 0 : S_j(t) \leq K_j\}$, $j = 1, 2$
  - R_j - recovery rate Asset j, $j = 1, 2$
- Motivation
  - Regulatory framework - Bilateral vs Unilateral
  - CCR not fully mitigated by collateral
  - $\approx$ 65% losses from CCR due to CVA during the financial crisis

Contribution

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- Structural approach to credit risk for unified treatment
  - CVA pricing
  - Right-Way-Risk/Wrong-Way-Risk
  - Mitigating clauses - netting & collateral
  - Gap Risk
  - Model Calibration
- Multivariate Lévy processes
  - Independent and stationary increments
  - Brownian motion with drift + pure jump process
  - Skewness and excess kurtosis
  - Joint evolution of risk factors
  - Improved calibration of credit spreads over short maturities Intermezzo 1
- Efficient numerical schemes (exotic option pricing)

Agenda

- Structural approach to credit risk modelling
- Multivariate Lévy processes
  - Construction
  - Dependence features
- CVA pricing
  - Model calibration
  - Correlation effect
  - Collateral
  - Gap risk
  - Netting
- Conclusions and work in progress

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Structural approach to default

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- Risk neutral dynamic

$$S_j(t) = S_j(0)e^{(r-q_j-\varphi_j(-i))t+X_j(t)}, \quad \forall j = 1, ..., n$$

- $X_j(t)$ - Lévy process
- $\varphi_j(-i)t$ characteristic exponent of $X_j(t)$
- $r > 0$ - risk free rate of interest
- $q_j > 0$ - dividend yield of the j^{th} asset
- "Non defaultable" underlying asset
- Dependence: factor representation (Ballotta and Bonfiglioli, 2014)

Multivariate Lévy processes

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- Lévy processes
  - Known characteristic function
  - Invariant under linear transformation
- $X_j(t) = Y_j(t) + a_j Z(t) \quad a_j \in \mathbb{R}, \forall j = 1, ..., n$
  - $X_j(t), Y_j(t), Z(t)$ are Lévy processes
  - $Y_j(t)$: idiosyncratic risk process
  - $Z(t)$: systematic risk process
  - $Y_j(t)$ and $Z(t)$ independent and distinct
  - Correlation coefficient correctly represents dependence

$$\rho_{jl} = a_j a_l \frac{\mathbb{V}ar(Z(1))}{\sqrt{\mathbb{V}ar(X_j(1))\mathbb{V}ar(X_l(1))}}$$

  - Dependence structure can be isolated from marginal distributions

CVA Pricing

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- Long position in a contract on S_3 (equity index, commodity, currency rate,...)
- First passage time approach

  Default event: $\tau_j = \inf\{t \geq 0 : S_j(t) \leq K_j\}$, $j = 1, 2$

- First to default problem

  $$CVA_1 = (1 - R_1)\mathbb{E}\left(1_{(\tau_1 \leq T)} 1_{(\tau_2 > \tau_1)} \Psi^+ (\tau_1; S_3, T)\right)$$

- 'Bucketing': default can only occur on time grid $\{t_j : 0 \leq j \leq N\}$ for $t_0 = 0, t_N = T$

  $$CVA_1 \approx (1 - R_1) \sum_{j=1}^{N} \mathbb{E}\left(1_{(t_{j-1} < \tau_1 \leq t_j)} 1_{(\tau_2 > t_j)} \Psi^+ (t_j; S_3, T)\right)$$

- Conditioning on $\{Z(t), 0 < t \leq T\}$

  $$CVA_1 \approx (1 - R_1) \sum_{j=1}^{N} \mathbb{E}\left[\mathbb{P}_Z (t_{j-1} < \tau_1 \leq t_j) \mathbb{P}_Z (\tau_2 > t_j) \mathbb{E}_Z \left(\Psi^+ (t_j; S_3, T)\right)\right]$$

Exposure: $\Psi^+(t) = D(t)v^+(t)$

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- Swap

  Payoff: $0 \leq t \leq T$ ($T_1, \cdots, T_{N_S} = T$: payment dates)

$$v^+(t) = \left(\sum_{i:T_i>t} \left(S_3(t)\, e^{-q_3(T_i-t)} - K_3 e^{-r(T_i-t)}\right)\right)^+$$
$$= \alpha(t,Z)\left(S_3(0)e^{(r-\varphi_{Y_3}(-i)t)+Y_3(t)} - K(t,Z)\right)^+$$

$$\alpha(t,Z) = e^{-\varphi_Z(-a_3 i)t + a_3 Z(t)} \sum_{i:T_i>t} e^{-q_3 T_i}$$
$$K(t,Z) = K_3 \sum_{i:T_i>t} e^{-r(T_i-t)}/\alpha(t,Z)$$

- Exposure: payoff of **European vanilla call option**
- Forward: set $N_S = 1$
- Standard Vanilla Option Pricing - COS method (Fang and Oosterlee, 2008)

Numerical implementation

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$$CVA_1 \approx (1-R_1)\sum_{i=1}^{N}\mathbb{E}\left[\mathbb{P}_Z\left(t_{i-1}<\tau_1\leq t_i\right)\mathbb{P}_Z\left(\tau_2>t_i\right)\mathbb{E}_Z\left(\Psi^+\left(t_i;S_3,T\right)\right)\right]$$

- $\tau_j = \inf\left\{t\geq 0 : Y_j(t)\leq \ln\frac{K_j}{S_j(0)}-(r-q_j-\varphi_j(-i))t-a_j Z(t)\right\}$
- Stochastic barrier due to common factor

Numerical implementation

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$$CVA_1 \approx (1 - R_1) \sum_{i=1}^{N} \mathbb{E}\left[\mathbb{P}_Z(t_{i-1} < \tau_1 \le t_i)\,\mathbb{P}_Z(\tau_2 > t_i)\,\mathbb{E}_Z\left(\Psi^+(t_i; S_3, T)\right)\right]$$

- $\tau_j = \inf\left\{t \ge 0 : Y_j(t) \le \ln \frac{K_j}{S_j(0)} - (r - q_j - \varphi_j(-i))t - a_j Z(t)\right\}$
- Stochastic barrier due to common factor
- Monte Carlo joint with Transform techniques: MC+Hillbert(P) - P: n. grid points
  - Monte Carlo: (M) trajectories of the commom component, Z
  - Hilbert Transform: conditional probabilities (Feng and Linetsky, 2008)
- Benchmark for efficiency test: (nested) Monte Carlo FullMC(k) - k: n. of nested iterations
- COS method: conditional option price (for all strikes) (Fang and Oosterlee, 2008)

Results I: Benchmark and efficiency

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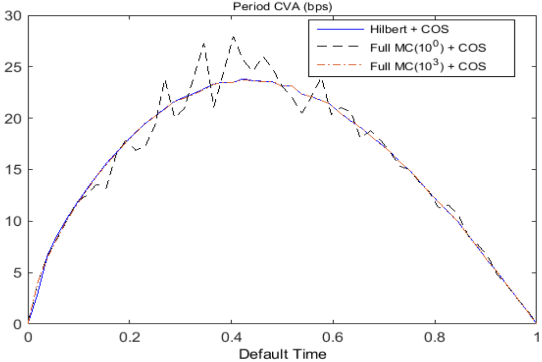

- M: 10^5; COS: 2^9 points $= P$, $L = 15$ (trunc. range)
- Efficiency index: $\sigma^2_{MC} t_{MC} / (\sigma^2_H t_H)$
  - 2.74 for $k = 1$ Monte Carlo nested iterations
  - 6.45 for $k = 10^3$ Monte Carlo nested iterations
  - NIG process

Market model: Example

- $X_j(t)$ - NIG process with parameters $(\theta_j, \sigma_j, k_j)$

  - $X_j(t) = \theta_j G_j(t) + \sigma_j W_j(G_j(t)) \quad \theta_j \in \mathbb{R}, \sigma_j \in \mathbb{R}^{++}$
  - $G_j(t)$ unbiased subordinator $IG(t/\sqrt{k_j}, 1/\sqrt{k_j})$, i.e.

    $$\mathbb{E}G_j(t) = t \qquad \mathbb{V}ar\left(G_j(t)\right) = k_j t$$

  - Characteristic exponent

    $$\varphi_j(u) = \frac{t}{k_j}\left(1 - \sqrt{1 - 2iu\theta_j k + u^2\sigma_j^2 k_j}\right)$$

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Market model: Example

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  - Characteristic exponent

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	θ_j	σ_j	k_j	RMSE	Std. Dev.	γ_1	γ_2
DB (S_1)	-0.22	0.25	0.55	1.33E-03	0.29	-1.09	2.48
ENI (S_2)	-0.18	0.10	0.34	1.29E-03	0.20	-0.87	1.14
BRENT (S_3)	0.07	0.19	0.08	2.18E-03	0.19	0.09	0.25

Calibration

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- $(\theta_j, \sigma_j, k_j)$ for $j = 1, 2, 3$
- Non linear least square fit
  - Default probabilities bootstrapped from CDS quotes (DB, ENI)
  - Option prices (Brent)

Market Data

- Term structure of interest rates bootstrapped using LIBOR and swap rates

Calibration

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- $(\theta_j, \sigma_j, k_j)$ for $j = 1, 2, 3$
- Non linear least square fit
  - Default probabilities bootstrapped from CDS quotes (DB, ENI)
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▸ Market Data

- Term structure of interest rates bootstrapped using LIBOR and swap rates
- Separation of margins from dependence ▸ Convolution
  - Lack of liquid products suitable for correlation calibration
- Correlation
  - Sensitivity analysis - Perturbation around sample correlation

▸ Intermezzo 2

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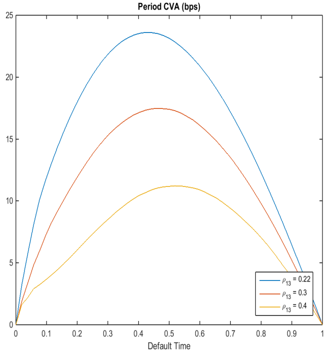
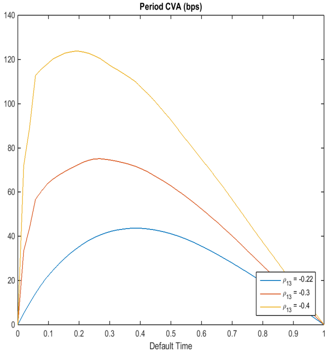
Results II: Swap





- $\rho_{13} > 0$: Right-Way-Risk $\rho_{13} < 0$: Wrong-Way-Risk
- T=1 year; $S_1(0) = S_2(0) = S_3(0) = 1$
- Weekly monitoring
- 10^6 Monte Carlo iterations, 2^{10} grid points
- Multiple cash flows product ("amortization" effect)

Forward

Collateral

- Risk mitigation tool
- Amount posted when (uncollateralized) exposure exceeds prespecified threshold
- Cash amount - no investment
- Minimum Transfer Amount (MTA): Amount below which no margin transfer is made
- It reduces frequency of collateral exchanges
- Notation
  - $E(t)$: uncollateralized exposure
  - H_1, H_2: thresholds (uni/bilateral)
  - M: Minimum Transfer Amount (MTA)
  - $C(t)$: collateral
  - $E_C(t)$: collateralized exposure
  - δt: margining period

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Collateral Pricing I: unilateral case

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- $H_1 > 0$: threshold for collateral posted by counterparty in investor's favour
- $C(t) = (E(t-\delta t) - H_1)^+$
- $E_C(t) = (E(t) - C(t))^+$

$$= \underbrace{E(t)}_{\text{Uncoll. Exp.}} - \underbrace{(E(t) - E_C(t))\, 1_{(C(t)>0)}}_{\text{Risk Mitigation due to collateral}}$$

- Alternative representation

$$E_C(t) = \underbrace{v^+(t) 1_{(v(t-\delta t)<H_1)}}_{\text{Correlation Gap call}} + \underbrace{(v(t) - v(t-\delta t) + H_1)^+ \, 1_{(v(t-\delta t)>H_1)}}_{\text{Calendar Spread call}}$$

- Numerics: condition on $v(t-\delta t)$

Collateral Pricing I: unilateral case

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- $H_1 > 0$: threshold for collateral posted by counterparty in investor's favour
- $C(t) = (E(t-\delta t) - H_1)^{+} \mathbb{1}_{(E(t-\delta t)-H_1>M)}$
- $E_C(t) = (E(t) - C(t))^{+}$

$$= \underbrace{E(t)}_{\text{Uncoll. Exp.}} - \underbrace{(E(t) - E_C(t))\, \mathbb{1}_{(C(t)>M)}}_{\text{Risk Mitigation due to collateral}}$$

- Alternative representation

$$E_C(t) = \underbrace{v^{+}(t)\mathbb{1}_{(v(t-\delta t)<H_1+M)}}_{\text{Correlation Gap call}} + \underbrace{(v(t) - v(t-\delta t) + H_1)^{+}\, \mathbb{1}_{(v(t-\delta t)>H_1+M)}}_{\text{Calendar Spread call}}$$

- Numerics: condition on $v(t-\delta t)$

Results III: EE Swap with Collateral

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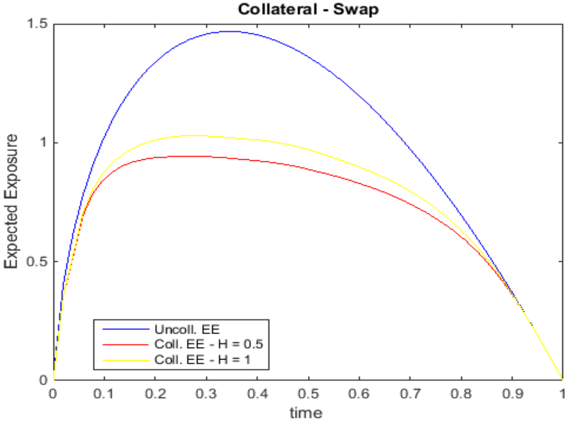

- % EE reduction:
- Swap: 27% ($H = 0.5$); 22% ($H = 1$)
- 2 weeks lag; base case ($\rho_{13} = 0.22$)
- Unilateral case

Collateral Pricing II: bilateral case

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- $H_2 < 0$: threshold for collateral posted by investor in counterparty's favour
- $C(t) = \underbrace{(v(t-\delta t) - H_1)^+}_{C^{(1)}(t)} + \underbrace{(v(t-\delta t) - H_2)^-}_{C^{(2)}(t)}$
- $E_C(t) = \underbrace{v^+(t)}_{\text{Uncoll. Exp.}} - \underbrace{\left(v(t) - E_C^{(1)}(t)\right) 1_{(C^{(1)}(t)>0)}}_{>0 \text{ (Risk Mitigation)}} - \underbrace{\left(v(t) - E_C^{(2)}(t)\right) 1_{(C^{(2)}(t)<0)}}_{<0 \text{ (Credit Exposure)}}$
- Alternative representation

$$E_C(t) = \underbrace{v^+(t) 1_{(H_2 < v(t-\delta t) < H_1)}}_{\text{Correlation Gap call}} + \underbrace{(v(t) - v(t-\delta t) + H_1)^+ 1_{(v(t-\delta t) > H_1)}}_{\text{Calendar Spread call}}$$
$$+ \underbrace{(v(t) - v(t-\delta t) + H_2)^+ 1_{(v(t-\delta t) < H_2)}}_{\text{Calendar Spread call}}$$

- MTA: similar to above

Results IV: bilateral case

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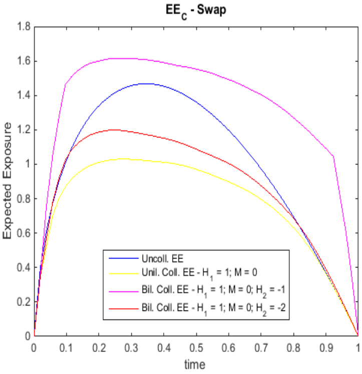

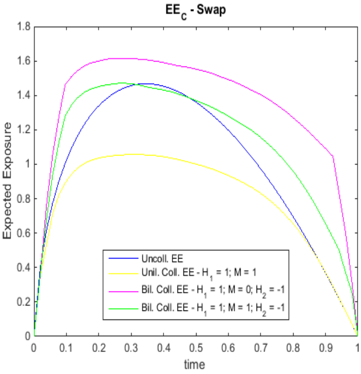

- 2 weeks lag; base case ($\rho_{13} = 0.22$)

Results V: Collateral vs W/RWR

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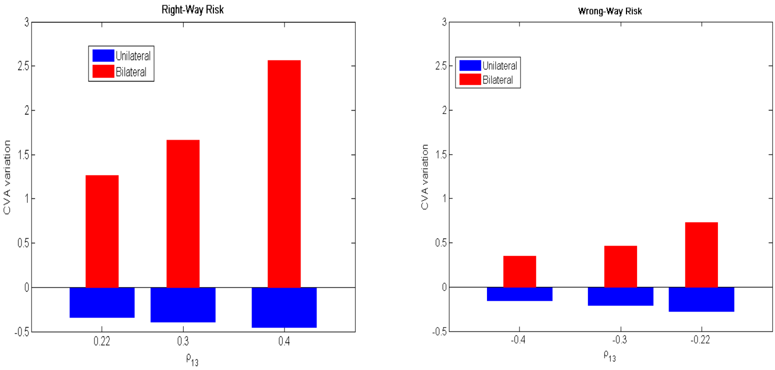

- Right/Wrong-Way Risk effect dependent on the collateral agreement
- $H_1 = 0.25; H_2 = 0; MTA = 0$

Quantifying Gap Risk

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- Gap Risk: counterparty default between margining dates and relevant adverse change in the exposure for the investor
- $\mathbb{P}\left(S_1(t) < K_1, S_2(t) > K_2, v(t) > 0 \middle| S_1(t_-) > K_1, S_2(t_-) > K_2, v(t_-) < 0\right)$

  $= \mathbb{P}\left(\Delta X_1(t) < -\varepsilon_1, \Delta X_2(t) > -\varepsilon_2, \Delta X_3(t) > \varepsilon_3\right)$

  $\approxeq \mathbb{P}\left(-a_1 \Delta Z(t) > \varepsilon_1, a_2 \Delta Z(t) > -\varepsilon_2, a_3 \Delta Z(t) > \varepsilon_3\right)$
- $v(t)$: contract value
- $\varepsilon_j = (r - q_j - \varphi_j(-i))\Delta t + \varDelta_j$ for $j = 1, 2$
- ε_3 defined according to contract type

  (example - Forward: $\varepsilon_3 = \varphi_3(-i)\Delta t + \varDelta_3$)
- $\varDelta_1, \varDelta_3 \uparrow \infty$, i.e. $\varepsilon_1, \varepsilon_3 \uparrow \infty$

Gap Risk (ctd)

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- $\rho_{13} < 0$ (otherwise probability is zero)

a) $\rho_{12} > 0$, $\rho_{23} < 0$

$$\begin{cases} \mathbb{P}\left(\max\left\{\frac{\varepsilon_1}{|a_1|}, \frac{\varepsilon_3}{a_3}\right\} < \Delta Z(t) < \frac{\varepsilon_2}{|a_2|}\right) & a_1, a_2 < 0 < a_3 \\ \mathbb{P}\left(-\frac{\varepsilon_2}{a_2} < \Delta Z(t) < \min\left\{-\frac{\varepsilon_1}{a_1}, -\frac{\varepsilon_3}{|a_3|}\right\}\right) & a_3 < 0 < a_1, a_2 \end{cases}$$

with

$$\begin{cases} \varepsilon_2 > |a_2| \max\left\{\frac{\varepsilon_1}{|a_1|}, \frac{\varepsilon_3}{a_3}\right\} & a_1, a_2 < 0 < a_3 \\ \varepsilon_2 > -a_2 \min\left\{-\frac{\varepsilon_1}{a_1}, -\frac{\varepsilon_3}{|a_3|}\right\} & a_3 < 0 < a_1, a_2 \end{cases}$$

b) $\rho_{12} < 0$, $\rho_{23} > 0$

$$\begin{cases} \mathbb{P}\left(\Delta Z(t) > \max\left\{\frac{\varepsilon_1}{|a_1|}, \frac{\varepsilon_3}{a_3}\right\}\right) & a_1 < 0 < a_2, a_3 \\ \mathbb{P}\left(\Delta Z(t) < \min\left\{-\frac{\varepsilon_1}{a_1}, -\frac{\varepsilon_3}{|a_3|}\right\}\right) & a_2, a_3 < 0 < a_1 \end{cases}$$

Results VI: Gap Risk - 2 weeks

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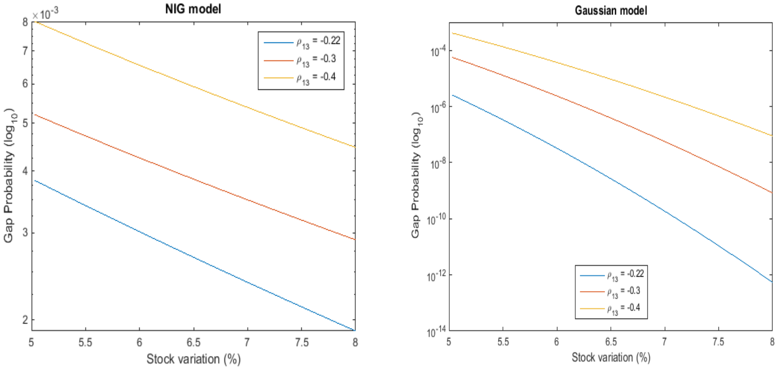

Scenarios

Netting

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- Upon default, losses are calculated at netted portfolio level
- $(\varpi_1, ..., \varpi_N)$ derivatives with discounted payoff Ψ_i

$$CVA_{1,W/0} = \sum_i^{N_b} \varpi_i \mathbb{E}\left[1_{(\tau_1 \leq \min(\tau_2, T))} \Psi_i^+\right]$$

$$CVA_{1,W} = \mathbb{E}\left[1_{(\tau_1 \leq \min(\tau_2, T))} \Pi^+\right] \qquad \Pi = \sum_i^{N_b} \varpi_i \Psi_i$$

Netting

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- Upon default, losses are calculated at netted portfolio level
- $(\varpi_1, ..., \varpi_N)$ derivatives with discounted payoff Ψ_i

$$CVA_{1,W/0} = \sum_i^{N_b} \varpi_i \mathbb{E}\left[1_{(\tau_1 \leq \min(\tau_2, T))} \Psi_i^+\right]$$

$$CVA_{1,W} = \mathbb{E}\left[1_{(\tau_1 \leq \min(\tau_2, T))} \Pi^+\right] \qquad \Pi = \sum_i^{N_b} \varpi_i \Psi_i$$

- A simple example
  - N_b swap contracts with maturity T and strike $K_j, j = 1, \cdots, N_b$
  - $\varpi_j = 1/N_b \ \forall j$
  - Payoff with netting

    $$\left(\mathcal{Y} - \sum_{l=3}^n w_l \bar{K}_l(t)\right)^+$$

    $$\mathcal{Y} = \sum_{l=3}^n \xi_l \qquad \xi_l = w_l \alpha_l(t, Z) S_l(0) e^{(r - q_l - \varphi_{Y_l}(-i))t + Y_l(t)}$$

    $$\bar{K}(t) = K_3 \sum_{j: T_j > t} e^{-r(T_j - t)}$$

  - 'homogeneous' copies of S_3

Netting: Numerics

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- Required: distribution of $\sum_{j=1}^{N_b} e^{-\varphi_{Y_j}(-i)t+Y_j(t)}$
- $Y_j(t)$: independent copies of the same (NIG) process
- Numerical methods
  - Exact: via convolution
  - Asymptotics: CLT implies

    $$\sum_{j=1}^{N_b} e^{-\varphi_{Y_j}(-i)t+Y_j(t)} \to \Phi\left(N_b, N_b\left(e^{-t(2\varphi_{Y_j}(-i)-\varphi_{Y_j}(-2i))} - 1\right)\right)$$
  - Barakat (1976) approximation

    $$\frac{1}{\sqrt{2\pi}} e^{-z^2/2}\left(1 + \frac{\gamma_1}{6N^{1/2}} h_3(z) + \frac{\gamma_2}{24N} h_4(z) + \frac{\gamma_1^2}{72N} h_6(z)\right)$$

    γ_1, γ_2: indices of skewness and excess kurtosis

    $h_k(z) = H_k(x)\phi(z)$, for $\phi(z)$ standard normal density (Edgeworth expansion)
- All info can be recovered from process Y (CF/pdf)

Results VII: Testing

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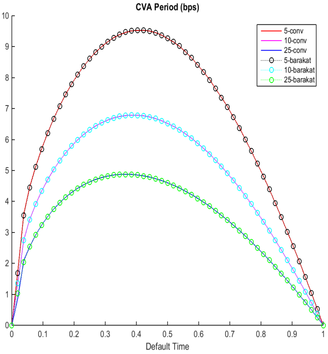

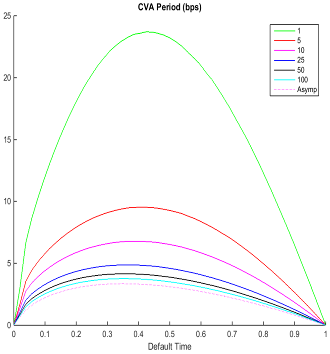

- Conv method becomes unstable for large N_b
- Barakat approximation works well also for small N_b

Results VIII: Netting & Diversification

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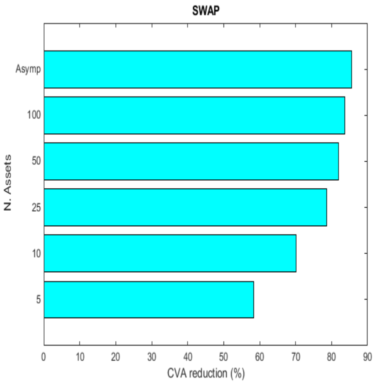

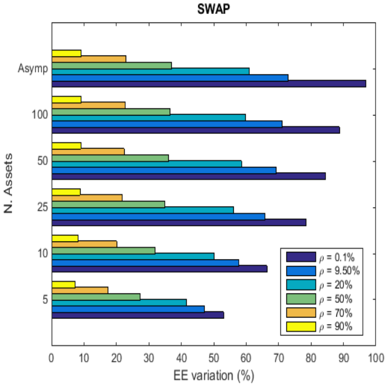

- Tot. CVA
- T=1 year; $S_j(0) = 1$ for $j = 1, \cdots, N_b$
- Weekly monitoring
- Base Case: $\rho = 9.50\%$

Conclusions & Work in progress

- Multivariate structural default model with jumps and dependent components
  - Unified treatment of CVA, Collateral, Netting
  - Integrated numerical scheme for pricing, calibration and correlation fitting
  - Sensitivity analysis with respect to relevant parameters
  - Impact of dependence on relevant measurements
- Mathematically and computationally tractable framework
- Work in progress
  - CVA with Netting and Collateral provisions (asymptotic results)
  - Default risk of the reference name (compound option problem) - defaultable CDS, vulnerable options
  - CVA of more complex instruments (interest rate derivatives)
  - ...

Synopsis

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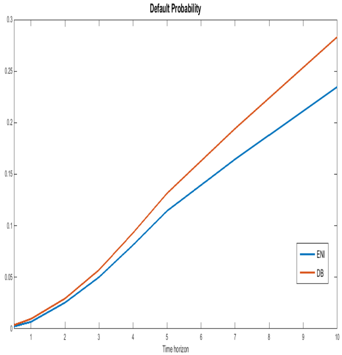

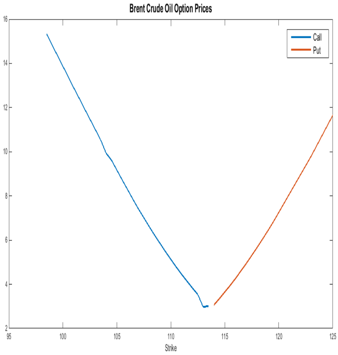

- CDS (DB, ENI) data source: Markit, June 26, 2014
- Default probabilities computed using Markit calculator
- Option (BRENT) data source: CME, June 26, 2014
- Settlement date: August 11, 2014

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Intermezzo 1: why jumps?

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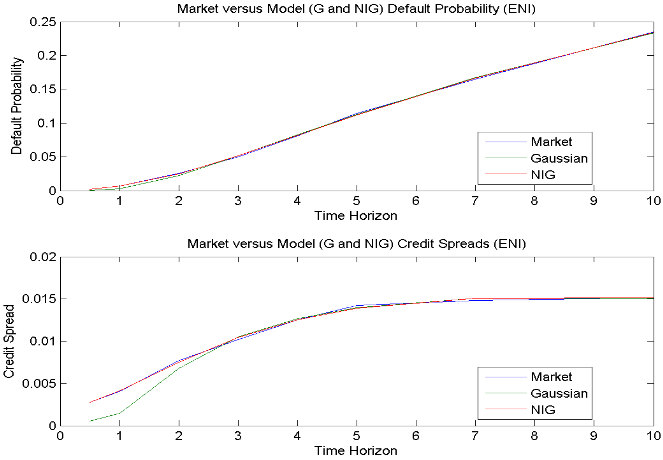

Intermezzo 1: why jumps?

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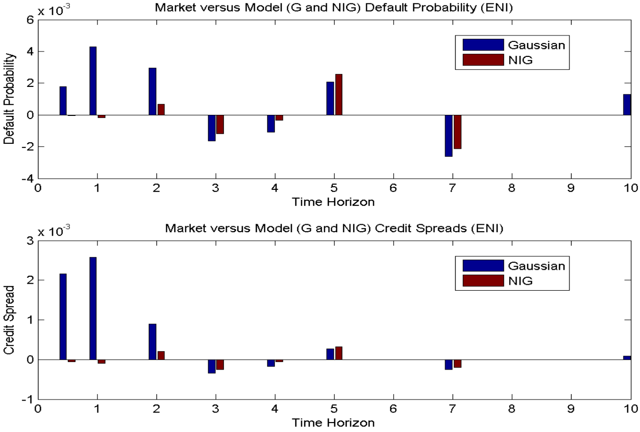

Back to Contribution   Back

Gap risk

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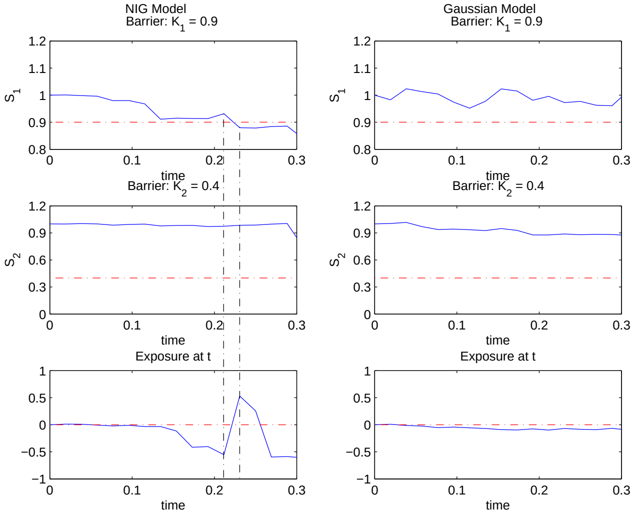

Back

At a glance

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- Any 1-d model per component
- Full range of dependencies
- Correlation coefficient correctly represents dependence

$$\rho_{jl} = a_j a_l \frac{\mathbb{V}ar(Z(1))}{\sqrt{\mathbb{V}ar(X_j(1))\mathbb{V}ar(X_l(1))}}$$

- Dimension of parameter set: linear in n (n. assets)
- Unified approach for all classes of Lévy processes
  - Subordinated Brownian motions: subordinator not "required"
  - JD processes: law of jump sizes depends on nature underlying shock
- Dependence structure can be isolated from marginal distributions

Convolution

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- $X = Y + aZ$ in distribution
- Given $\mathbf{X}$, choose Z parameters s.t.

$$\min \sum_{j=1}^{3} \int \left|\phi_{X_j}(u) - \phi_{Y_j}(u)\phi_Z(a_j u)\right|^2 du, \quad j = 1, 2, 3$$

- Choose $\mathbf{a}$ to fit given correlation matrix
- Obtain $\mathbf{Y}$ parameters s.t.
  $c_k^{X_j} = c_k^{Y_j} + a_j^k c_k^Z, \quad j = 1, 2, 3; k = 1, \cdots, 4$

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Convolution

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- Correlation matrix: sample correlation (2 years daily observations)

DB	1		
ENI	0.65	1	
BRENT	0.22	0.29	1

Convolution: Fitting error

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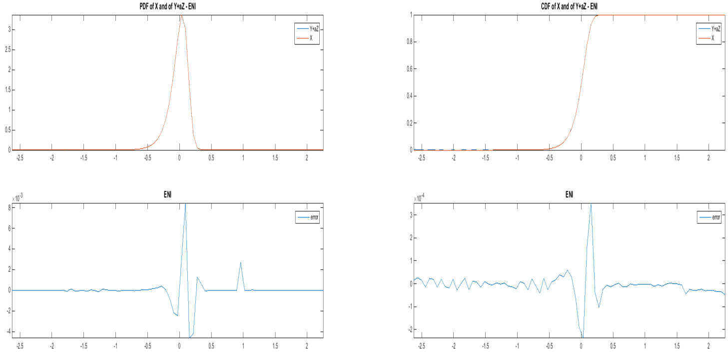

Back

Intermezzo 2

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- Right-way risk: the exposure tends to decrease when the counteparty credit quality deteriorates
- Firm 1 credit quality worsens, call option on S_3 moves out of the money
- $\rho_{13}^{\mathbf{X}} > 0$
- Wrong-way risk: the exposure tends to increase when the counteparty credit quality deteriorates
- Firm 1 credit quality worsens, call option on S_3 moves in the money
- $\rho_{13}^{\mathbf{X}} < 0$

Back

CVA: Forward

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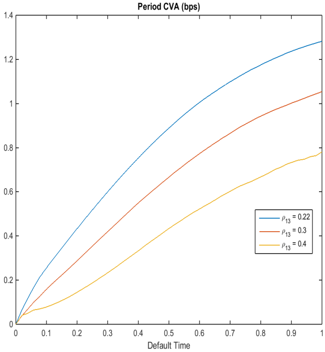

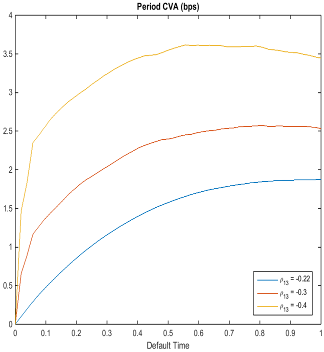

- $\rho_{13} > 0$: Right-Way-Risk $\qquad$ $\rho_{13} < 0$: Wrong-Way-Risk
- T=1 year; $S_1(0) = S_2(0) = S_3(0) = 1$
- Weekly monitoring
- 10^6 Monte Carlo iterations, 2^{10} grid points
- Single cash flow product ("diffusion" effect)

Back

Impact of collateral (unilateral)

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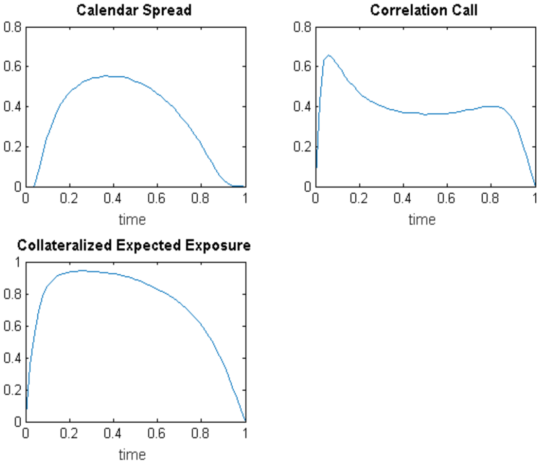

- Swap contract; 2 weeks lag; base case ($\rho_{13} = 0.22$)

Collateral and MTA: EE Swap

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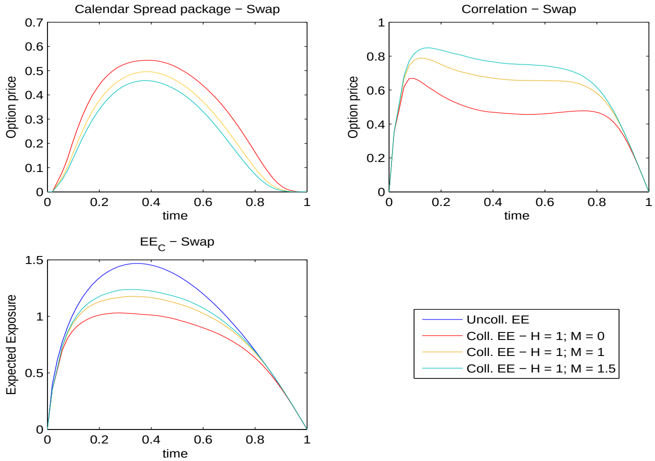
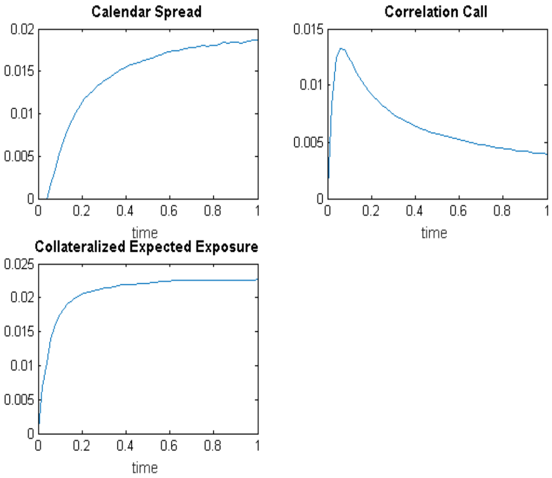


- % EE reduction:
- Swap: 22% ($H = 1, M = 0$); 6.6% ($H = 1, M = 1$); 2% ($H = 1, M = 1.5$);
- 2 weeks lag; base case ($\rho_{13} = 0.22$)
- Unilateral case

Expected Exposure with collateral I



- Forward contract; 2 weeks lag; base case ($\rho_{13} = 0.22$)

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Collateral Pricing II: bilateral case

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- $H_2 < 0$: collateral posted by investor in counterparty's favour
- $C(t) = \underbrace{(v(t-\delta t) - H_1)^+}_{C^{(1)}(t)} 1_{(v(t-\delta t)-H_1>M)} + \underbrace{(v(t-\delta t) - H_2)^-}_{C^{(2)}(t)} 1_{(v(t-\delta t)-H_2<-M)}$
- $E_C(t) = \underbrace{v^+(t)}_{\text{Uncoll. Exp.}} - \underbrace{\left(v(t) - E_C^{(1)}(t)\right) 1_{(C^{(1)}(t)>0)}}_{>0 \text{ (Risk Mitigation)}} - \underbrace{\left(v(t) - E_C^{(2)}(t)\right) 1_{(C^{(2)}(t)<0)}}_{<0 \text{ (Credit Exposure)}}$
- Alternative representation

$$E_C(t) = \underbrace{v^+(t)1_{(H_2-M<v(t-\delta t)<H_1+M)}}_{\text{Correlation Gap call}} + \underbrace{(v(t) - v(t-\delta t) + H_1)^+ 1_{(v(t-\delta t)>H_1+M)}}_{\text{Calendar Spread call}}$$

$$+ \underbrace{(v(t) - v(t-\delta t) + H_2)^+ 1_{(v(t-\delta t)<H_2-M)}}_{\text{Calendar Spread call}}$$

NIG vs Brownian motion: the tails

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- NIG distribution
  - Upper tail

$$\mathbb{P}(X > x) \approx 2C\frac{e^{-\lambda_U x}}{\sqrt{x}} - 2C\sqrt{\lambda_U \pi}\,\mathrm{erfc}\left(\sqrt{\lambda_U x}\right) \qquad x > 0$$

  - Lower tail

$$\mathbb{P}(X < -x) \approx 2C\frac{e^{-\lambda_L x}}{\sqrt{x}} - 2C\sqrt{\lambda_L \pi}\,\mathrm{erfc}\left(\sqrt{\lambda_L x}\right) \qquad x > 0$$

- Gaussian distribution
  - Upper tail

$$\mathbb{P}(X > x) = \frac{1}{2}\,\mathrm{erfc}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)$$

  - Lower tail

$$\mathbb{P}(X \leq x) = \frac{1}{2} + \frac{1}{2}\,\mathrm{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)$$