STATISTICAL METHODS IN FINANCE, ASSIGNMENT 1

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Solution (Exercise 1: Skewness and kurtosis)**.**

(i) *From Hölder's inequality (Proposition 2.2.3 in the notes), Hölder's inequality reads*

$$\mathbb{E}[|XY|] \leq \mathbb{E}\left[|X|^p\right]^{1/p} \mathbb{E}\left[|X|^q\right]^{1/q},$$

for every $p \in (1, \infty)$ and q such that $p^{-1} + q^{-1} = 1$, whenever all expectations are finite. Take $Y = 1$ and $X = Z^p$ almost surely, then

$$\mathbb{E}\left[|Z|^p\right] \leq \mathbb{E}\left[|Z|^{rp}\right]^{1/p};$$

Setting $q := rp$ then yields $\mathbb{E}\left[|Z|^p\right] \leq \mathbb{E}\left[|Z|^q\right]^{r/q}$, and it is easy to check that $r^{-1} + p^{-1} = 1$.

(ii) *If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $(X - \mu)/\sigma \sim \mathcal{N}(0, 1)$. We can then compute by direct integration against the Gaussian density:*

$$\mathcal{S} := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x^3 \exp\left\{-\frac{x^2}{2}\right\} \mathrm{d}x = 0,$$

since the integrand is an odd function. Similarly, by integration by parts,

$$\kappa := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x^4 \exp\left\{-\frac{x^2}{2}\right\} \mathrm{d}x = \frac{1}{5\sqrt{2\pi}} \left[x^5 \exp\left\{-\frac{x^2}{2}\right\}\right]_{\mathbb{R}} + \frac{1}{5\sqrt{2\pi}} \int_{\mathbb{R}} x^6 \exp\left\{-\frac{x^2}{2}\right\} \mathrm{d}x.$$

Denoting by I_4, the kurtosis, we note that the previous equality can be rewrittent as $I_4 = \frac{1}{5} I_6$. Continuing the recursion backwards, this yields

$$I_6 = 5I_4 = 5 \cdot 3I_2 = 5 \cdot 3I_0,$$

with clearly $I_0 = 1$, and therefore $I_4 = \kappa = 3$.

(iii) *If $X \sim \mathcal{U}_{[a,b]}$, then, $\mathbb{E}[X] = (b - a)/2$ and $\mathbb{V}[X] = (b - a)^2/12$. Again, since the distribution of X is symmetric around its mean, then the skewness is equal to zero (the function $x \mapsto x^3$ being odd). Regarding the kurtosis, taking for simplicity $a = 0$ and $b = 1$, we can write*

$$\kappa = \int_{[0,1]} x^4 \mathrm{d}x = \frac{1}{5}.$$

(iv) *If $X \sim \mathcal{E}(\lambda)$, then, by integration by parts,*

$$\mathbb{E}[X] = \int_{[0,\infty)} \lambda x \mathrm{e}^{-\lambda x} \mathrm{d}x = -\left[x \mathrm{e}^{-\lambda x}\right]_{[0,\infty)} + \int_{[0,\infty)} \mathrm{e}^{-\lambda x} \mathrm{d}x = \frac{1}{\lambda}.$$

Likewise,

$$\mathbb{E}[X^2] = \lambda \int_{[0,\infty)} x^2 \mathrm{e}^{-\lambda x} \mathrm{d}x = -\left[x^2 \mathrm{e}^{-\lambda x}\right]_{[0,\infty)} + 2 \int_{[0,\infty)} x \mathrm{e}^{-\lambda x} \mathrm{d}x = \frac{2}{\lambda} \mathbb{E}[X],$$

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and hence

$$\mathbb{V}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{2}{\lambda}\mathbb{E}[X] - \mathbb{E}[X]^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$$

Computations for the skewness and the skewness and the kurtosis are similar, and we obtain $\mathcal{S} = 2$ *and* $\kappa = 9$.

(v) *See the IPython notebook.*

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Solution (Exercise 2: Convergence and Central Limit Theorem)**.**

(i) *Since* $X \sim Poisson(\lambda)$*, we can compute directly*

$$\mathbb{E}[X] = \sum_{k\geq 0} k\frac{\lambda^k \mathrm{e}^{-\lambda}}{k!} = \mathrm{e}^{-\lambda}\lambda\sum_{k\geq 1}\frac{\lambda^{k-1}}{(k-1)!} = \mathrm{e}^{-\lambda}\lambda\mathrm{e}^{\lambda} = \lambda,$$

$$\mathbb{E}[X^2] = \sum_{k\geq 0} k^2\frac{\lambda^k \mathrm{e}^{-\lambda}}{k!} = \mathrm{e}^{-\lambda}\sum_{k\geq 1}k\frac{\lambda^k}{(k-1)!} = \mathrm{e}^{-\lambda}\left(\sum_{k\geq 1}(k-1)\frac{\lambda^k}{(k-1)!} + \sum_{k\geq 1}\frac{\lambda^k}{(k-1)!}\right)$$

$$= \mathrm{e}^{-\lambda}\lambda^2\sum_{k\geq 2}\frac{\lambda^{k-2}}{(k-2)!} + \mathrm{e}^{-\lambda}\lambda\sum_{k\geq 1}\frac{\lambda^{k-1}}{(k-1)!} = \lambda^2 + \lambda,$$

and therefore $\mathbb{V}(X) = \lambda$.

(ii) *Recall that* $S_n = \frac{1}{n}\sum_{k=1}^{n} X_i$. *The sequence* $(X_k)_{k\in\mathbb{N}}$*'s is iid and integrable, and therefore the weak law of large numbers implies that* $(S_n)_{n\in\mathbb{N}}$ *converges to* λ *in probability.*

(iii) *Let* $T_n := \exp(-S_n)$*. For any fixed* $\varepsilon > 0$,

$$\begin{aligned}\mathbb{P}\left(\left|T_n - \mathrm{e}^{-\lambda}\right| < \varepsilon\right) &= \mathbb{P}\left(\mathrm{e}^{-\lambda} - \varepsilon < T_n < \mathrm{e}^{-\lambda} + \varepsilon\right) \\ &= \mathbb{P}\left(-\ln(\mathrm{e}^{-\lambda} + \varepsilon) < S_n < -\ln(\mathrm{e}^{-\lambda} - \varepsilon)\right) \\ &= \mathbb{P}\left(\lambda - \ln(1 + \varepsilon\mathrm{e}^{\lambda}) < S_n < \lambda - \ln(1 - \varepsilon\mathrm{e}^{\lambda})\right),\end{aligned}$$

which converges to 1 *as* n *tends to infinity, since the sequence* (S_n) *converges in probability.*

(iv) *Let* $x > 0$*. The Central Limit Theorem implies that*

$$\lim_{n\uparrow\infty}\frac{S_n - \lambda}{\sqrt{\lambda/n}} = Z \sim \mathcal{N}(0,1) \qquad \textit{in probability.}$$

Therefore,

$$\mathbb{P}(T_n \leq x) = \mathbb{P}\left(S_n \geq -\ln(x)\right) = \mathbb{P}\left(\sqrt{n}\frac{S_n - \lambda}{\sqrt{\lambda}} \geq \sqrt{n}\frac{-\ln(x) - \lambda}{\sqrt{\lambda}}\right).$$

If $x < \mathrm{e}^{-\lambda}$ *then* $-\ln(x) - \lambda > 0$ *and thus* $\sqrt{n}\frac{-\ln(x)-\lambda}{\sqrt{\lambda}}$ *diverges to* $+\infty$*. Conversely if* $x > \mathrm{e}^{-\lambda}$ *then* $\sqrt{n}\frac{-\ln(x)-\lambda}{\sqrt{\lambda}}$ *diverges to* $-\infty$*. Hence,*

$$\lim_{n\uparrow\infty}\mathbb{P}(T_n \leq x) = \mathbb{P}(Z \geq \lim_{n\to\infty}\sqrt{n}\frac{-\ln(x) - \lambda}{\sqrt{\lambda}}) = \begin{cases}\mathbb{P}(Z \geq +\infty) = 0, & \textit{if } x < \mathrm{e}^{-\lambda},\\ \mathbb{P}(Z \geq -\infty) = 1, & \textit{if } x > \mathrm{e}^{-\lambda}.\end{cases}$$

Solution (Exercise 3: Convergence of random variables)**.**

(i) *Straightforward from the lecture notes.*

(ii) *We first prove the claim:*

$$\begin{aligned}\mathbb{P}(Y \leq x) &= \mathbb{P}(Y \leq x, X \leq x+\varepsilon) + \mathbb{P}(Y \leq x, X > x+\varepsilon)\\ &\leq \mathbb{P}(X \leq x+\varepsilon) + \mathbb{P}(Y - X \leq x - X, x - X < -\varepsilon)\\ &\leq \mathbb{P}(X \leq x+\varepsilon) + \mathbb{P}(Y - X < -\varepsilon)\\ &\leq \mathbb{P}(X \leq x+\varepsilon) + \mathbb{P}(Y - X < -\varepsilon) + \mathbb{P}(Y - X > \varepsilon)\\ &\leq \mathbb{P}(X \leq x+\varepsilon) + \mathbb{P}(|Y - X| > \varepsilon).\end{aligned}$$

We now move on to the general proof. We need to show pointwise convergence of the cdf at every point of continuity. Let F be the limiting cdf, and x such a point. For any $\varepsilon > 0$, the claim yields

$$\mathbb{P}(X_n \leq x) \leq \mathbb{P}(X \leq x+\varepsilon) + \mathbb{P}(|X_n - X| > \varepsilon) \qquad \text{and} \qquad \mathbb{P}(X \leq x-\varepsilon) \leq \mathbb{P}(X_n \leq x) + \mathbb{P}(|X_n - X| > \varepsilon),$$

so that

$$\mathbb{P}(X \leq x-\varepsilon) - \mathbb{P}(|X_n - X| > \varepsilon) \leq \mathbb{P}(X_n \leq x) \leq \mathbb{P}(X \leq x+\varepsilon) + \mathbb{P}(|X_n - X| > \varepsilon).$$

Taking the limit as n tends to infinity yields

$$F(x-\varepsilon) \leq \lim_{n\uparrow\infty} \mathbb{P}(X_n \leq x) \leq F(x+\varepsilon),$$

and the result follows since x is a continuity point of F.

(iii) *Let c be the constant to which the sequence converges in distribution, and fix $\varepsilon > 0$. Then $\mathbb{P}(|X_n - c| \geq \varepsilon) = \mathbb{P}(X_n \notin \mathcal{B}_\varepsilon(c))$, where $\mathcal{B}_\varepsilon(c)$ denotes the ball of radius ε centred at the point c. Therefore, convergence in distribution implies that*

$$\lim_{n\uparrow\infty} \mathbb{P}(|X_n - c| \geq \varepsilon) \leq \limsup_{n\uparrow\infty} \mathbb{P}(|X_n - c| \geq \varepsilon) \leq \limsup_{n\uparrow\infty} \mathbb{P}(X_n \notin \mathcal{B}_\varepsilon(c)) \leq \mathbb{P}(c \notin \mathcal{B}_\varepsilon(c)) = 0,$$

which is exactly convergence in probability.

Solution (Exercise 4: Joint distributions)**.**

(i) *Denote by $\boldsymbol{X} = (X_1, X_2)$ and $\boldsymbol{Y} = (Y_1, Y_2)$. We can rewrite the definition of $\boldsymbol{Y}$ as $\boldsymbol{Y} = \boldsymbol{\mu} + \Sigma\boldsymbol{X}$, where the matrix Σ reads*

$$\Sigma = \begin{pmatrix} \sigma_1\overline{\rho} & \rho\sigma_1 \\ 0 & \sigma_2 \end{pmatrix},$$

where we denote $\overline{\rho} := \sqrt{1-\rho^2} \in (0,1)$. Since both σ_1 and σ_2 are strictly positive, the matrix Σ is invertible, and we can write $\boldsymbol{X} = \Sigma^{-1}(\boldsymbol{Y} - \boldsymbol{\mu})$, where

$$\Sigma^{-1} = \begin{pmatrix} \frac{1}{\sigma_1\overline{\rho}} & \frac{-\rho}{\sigma_2\overline{\rho}} \\ 0 & \frac{1}{\sigma_2} \end{pmatrix} \tag{0.1}$$

The Jacobian then reads

$$J(y_1, y_2) = \left|\Sigma^{-1}\right| = \frac{1}{\sigma_1\sigma_2\overline{\rho}}.$$

The joint density of $\boldsymbol{Y}$ then reads

$$f_{\boldsymbol{Y}}(\boldsymbol{y}) = f_{\boldsymbol{X}}(\boldsymbol{x})|J(\boldsymbol{y})| = \frac{1}{2\pi}\exp\left\{-\frac{x_1^2 + x_2^2}{2}\right\} J(\boldsymbol{y}),$$

where $\boldsymbol{x}^\top = \Sigma^{-1}\boldsymbol{y}^\top$. *More explicitly, we can compute, from* (0.1),

$$x_1 = \frac{1}{\overline{\rho}}\left(\frac{y_1-\mu_1}{\sigma_1} - \frac{\rho(y_2-\mu_2)}{\sigma_2}\right) \qquad \text{and} \qquad x_2 = \frac{y_2-\mu_2}{\sigma_2},$$

so that

$$\begin{aligned}
x_1^2 + x_2^2 &= \left[\frac{1}{\overline{\rho}}\left(\frac{y_1-\mu_1}{\sigma_1} - \frac{\rho(y_2-\mu_2)}{\sigma_2}\right)\right]^2 + \left(\frac{y_2-\mu_2}{\sigma_2}\right)^2\\
&= \frac{1}{1-\rho^2}\left[\frac{(y_1-\mu_1)^2}{\sigma_1^2} + \frac{\rho^2(y_2-\mu_2)^2}{\sigma_2^2} - 2\rho\frac{(y_1-\mu_1)(y_2-\mu_2)}{\sigma_1\sigma_2} + \left(1-\rho^2\right)\frac{(y_2-\mu_2)^2}{\sigma_2^2}\right]\\
&= \frac{1}{1-\rho^2}\left[\frac{(y_1-\mu_1)^2}{\sigma_1^2} + \frac{(y_2-\mu_2)^2}{\sigma_2^2} - 2\rho\frac{(y_1-\mu_1)(y_2-\mu_2)}{\sigma_1\sigma_2}\right],
\end{aligned}$$

and therefore

$$\begin{aligned}
f_{\boldsymbol{Y}}(\boldsymbol{y}) &= \frac{1}{2\pi}\exp\left\{-\frac{x_1^2+x_2^2}{2}\right\} J(\boldsymbol{y})\\
&= \frac{1}{2\pi\sigma_1\sigma_2\overline{\rho}}\exp\left\{-\frac{1}{2\left(1-\rho^2\right)}\left[\frac{(y_1-\mu_1)^2}{\sigma_1^2} + \frac{(y_2-\mu_2)^2}{\sigma_2^2} - 2\rho\frac{(y_1-\mu_1)(y_2-\mu_2)}{\sigma_1\sigma_2}\right]\right\}\\
&\frac{1}{2\pi|\Sigma|^{1/2}}\exp\left\{-\frac{1}{2}(\boldsymbol{y}-\boldsymbol{\mu})^\top\Sigma^{-1}(\boldsymbol{y}-\boldsymbol{\mu})\right\}, \qquad \text{for all } \boldsymbol{y}\in\mathbb{R}^2.
\end{aligned}$$

To compute the marginal densities of Y_1 *and* Y_2*, we need to integrate out the joint density:*

$$\begin{aligned}
f_{Y_1}(y_1) &= \int_{\mathbb{R}} f_{\boldsymbol{Y}}(y_1,y_2)\mathrm{d}y_2\\
&= \frac{1}{2\pi\sigma_1\sigma_2\overline{\rho}}\int_{\mathbb{R}}\exp\left\{-\frac{1}{2\left(1-\rho^2\right)}\left[\frac{(y_1-\mu_1)^2}{\sigma_1^2} + \frac{(y_2-\mu_2)^2}{\sigma_2^2} - 2\rho\frac{(y_1-\mu_1)(y_2-\mu_2)}{\sigma_1\sigma_2}\right]\right\}\mathrm{d}y_2\\
&= \frac{1}{2\pi\sigma_1\overline{\rho}}\int_{\mathbb{R}}\exp\left\{-\frac{z_1^2+z_2^2-2\rho z_1 z_2}{2\left(1-\rho^2\right)}\right\}\mathrm{d}z_2\\
&= \frac{1}{2\pi\sigma_1\overline{\rho}}\int_{\mathbb{R}}\exp\left\{-\frac{(z_2-\rho z_1)^2+\left(1-\rho^2\right)z_1^2}{2\left(1-\rho^2\right)}\right\}\mathrm{d}z_2\\
&= \frac{\exp\left\{-\frac{z_1^2}{2}\right\}}{2\pi\sigma_1\overline{\rho}}\int_{\mathbb{R}}\exp\left\{-\frac{(z_2-\rho z_1)^2}{2\left(1-\rho^2\right)}\right\}\mathrm{d}z_2 = \frac{1}{\sigma_1\sqrt{2\pi}}\exp\left\{-\frac{z_1^2}{2}\right\} = \frac{1}{\sigma_1\sqrt{2\pi}}\exp\left\{-\frac{(y_1-\mu_1)^2}{2\sigma_1^2}\right\},
\end{aligned}$$

where we set $z_1 := (y_1-\mu_2)/\sigma_1$ *and* $z_2 := (y_2-\mu_2)/\sigma_2$ *in the third line. Hence* Y_1 *is Gaussian with mean* μ_1 *and variance* σ_1^2*. The marginal distribution of* Y_2 *follows analogous computations. Regarding the* *conditional densities, we have, using the previous results,*

$$
\begin{aligned}
f_{Y_1|Y_2}(y_1|y_2) &= \frac{f_{Y_1,Y_2}(y_1,y_2)}{f_{Y_2}(y_2)} \\
&= \frac{\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(y_1-\mu_1)^2}{\sigma_1^2}+\frac{(y_2-\mu_2)^2}{\sigma_2^2}-2\rho\frac{(y_1-\mu_1)(y_2-\mu_2)}{\sigma_1\sigma_2}\right]\right\}}{2\pi\sigma_1\sigma_2\overline{\rho}} \frac{1}{\frac{1}{\sigma_2\sqrt{2\pi}}\exp\left\{-\frac{(y_2-\mu_2)^2}{2\sigma_2^2}\right\}} \\
&= \frac{1}{\sigma_1\overline{\rho}\sqrt{2\pi}}\exp\left\{-\frac{1}{1-\rho^2}\left[\frac{(y_1-\mu_1)^2}{2\sigma_1^2}+\rho^2\frac{(y_2-\mu_2)^2}{2\sigma_2^2}-\rho\frac{(y_1-\mu_1)(y_2-\mu_2)}{\sigma_1\sigma_2}\right]\right\} \\
&= \frac{1}{\sigma_1\overline{\rho}\sqrt{2\pi}}\exp\left\{-\frac{1}{2\sigma_1^2(1-\rho^2)}\left[(y_1-\mu_1)^2+\rho^2\sigma_1^2\frac{(y_2-\mu_2)^2}{\sigma_2^2}-2\rho\sigma_1\frac{(y_1-\mu_1)(y_2-\mu_2)}{\sigma_2}\right]\right\} \\
&= \frac{1}{\sigma_1\overline{\rho}\sqrt{2\pi}}\exp\left\{-\frac{1}{2\sigma_1^2(1-\rho^2)}\left[(y_1-\mu_1)^2+\widetilde{y}_2^2-2(y_1-\mu_1)\widetilde{y}_2\right]\right\} \\
&= \frac{1}{\sigma_1\overline{\rho}\sqrt{2\pi}}\exp\left\{-\frac{1}{2\sigma_1^2(1-\rho^2)}\left[y_1^2-2(\mu_1+\widetilde{y}_2)y_1+\mu_1^2+\widetilde{y}_2^2+2\mu_1\widetilde{y}_2\right]\right\} \\
&= \frac{1}{\sigma_1\overline{\rho}\sqrt{2\pi}}\exp\left\{-\frac{1}{2\sigma_1^2(1-\rho^2)}\left[y_1-(\mu_1+\widetilde{y}_2)^2\right]\right\},
\end{aligned}
$$

with $\widetilde{y}_2 := \rho\sigma_1(y_2-\mu_2)/\sigma_2$ *and* $\widetilde{\mu} := \mu_1+\rho\sigma_1(y_2-\mu_2)/\sigma_2$. *Therefore* $Y_1|Y_2$ *is also Gaussian with mean* $\mu_1+\widetilde{y}_2$ *and variance* $\sigma_1^2(1-\rho^2)$.

In order to compute the correlation, we first compute, using the tower property for expectations,

$$
\begin{aligned}
\mathbb{E}[Y_1,Y_2] &= \mathbb{E}\left[\mathbb{E}\left[Y_1Y_2|Y_2\right]\right] \\
&= \mathbb{E}\left[\left(\mu_1+\frac{\rho\sigma_1}{\sigma_2}(Y_2-\mu_2)\right)Y_2\right] \\
&= \left(\mu_1-\frac{\rho\sigma_1\mu_2}{\sigma_2}\right)\mathbb{E}[Y_2]+\frac{\rho\sigma_1}{\sigma_2}\mathbb{E}\left[Y_2^2\right] \\
&= \left(\mu_1-\frac{\rho\sigma_1\mu_2}{\sigma_2}\right)\mu_2+\frac{\rho\sigma_1}{\sigma_2}\left(\sigma_2^2+\mu_2^2\right) = \mu_1\mu_2+\rho\sigma_1\sigma_2,
\end{aligned}
$$

and we therefore deduce

$$
\mathrm{Cov}[Y_1,Y_2] = \mathbb{E}[Y_1,Y_2]-\mathbb{E}[Y_1]\mathbb{E}[Y_2] = \mu_1\mu_2+\rho\sigma_1\sigma_2-\mu_1\mu_2 = \rho\sigma_1\sigma_2,
$$

and

$$
\mathrm{Corr}[Y_1,Y_2] = \frac{\mathrm{Cov}[Y_1,Y_2]}{\sqrt{\mathbb{V}[Y_1]\mathbb{V}[Y_2]}} = \rho.
$$

(ii) Considering now the second problem, we can write the inverse transformation $(U_1,U_2) = \varphi(X_1,X_2) = (\varphi_1((X_1,X_2),\varphi_2(X_1,X_2))$, *with*

$$
U_1 = \varphi_1(X_1,X_2) = \exp\left\{-\frac{X_1^2+X_2^2}{2}\right\} \qquad \text{and} \qquad U_2 = \varphi_2(X_1,X_2) = \frac{1}{2\pi}\arctan\left(\frac{X_2}{X_1}\right).
$$

The Jacobian of φ *now reads*

$$
J_\varphi(x_1,x_2) := \begin{vmatrix}\partial_{x_1}\varphi_1 & \partial_{x_2}\varphi_1 \\ \partial_{x_1}\varphi_2 & \partial_{x_2}\varphi_2\end{vmatrix} = \begin{vmatrix} x_1\exp\left\{-\frac{x_1^2+x_2^2}{2}\right\} & x_2\exp\left\{-\frac{x_1^2+x_2^2}{2}\right\} \\ -\frac{1}{2\pi}\frac{x_2}{x_1^2+x_2^2} & \frac{1}{2\pi}\frac{x_1}{x_1^2+x_2^2}\end{vmatrix} = \frac{1}{2\pi}\exp\left\{-\frac{x_1^2+x_2^2}{2}\right\}.
$$

The joint density can therefore be computed as

$$f_{X_1,X_2}(x_1,x_2) = f_{U_1,U_2}\left(\exp\left\{-\frac{x_1^2+x_2^2}{2}\right\}, \frac{1}{2\pi}\arctan\left(\frac{x_2}{x_1}\right)\right) J(x_1,x_2) = \frac{1}{2\pi}\exp\left\{-\frac{x_1^2+x_2^2}{2}\right\}.$$

Since, for any $(x_1,x_2)\in\mathbb{R}^2)$, $f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1)f_{X_2}(x_2)$, with

$$f_{X_i}(x_i) = \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{x_i^2}{2}\right\}, \qquad \textit{for } i=1,2,$$

then X_1 and X_2 are two independent centered Gaussian random variables with unit variance.

Exercise 1 (Log-normal distribution)**.** The two questions below are independent. Consider the standard Gaussian distribution $X \sim \mathcal{N}(0,1)$, with density

$$f(x) = \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{x^2}{2}\right\}, \qquad \text{for all } x\in\mathbb{R}.$$

(i) Compute $\mathbb{E}[X]$, $\mathbb{V}[X]$ and $\mathbb{E}\left[\mathrm{e}^{uX}\right]$ for all $u\in\mathbb{R}$ such that the expectation is well defined.
(ii) Define $Y := \exp\{X\}$. Compute its density, expectation, variance and moment generating function.
(iii) Does Y have a symmetric distribution? Compute its skewness to confirm your guess.

Solution (Log-normal distribution)**.**

(i) The density being symmetric with respect to the origin, the expectation is null, and, using the Solution to Exercise 1, we have $I_2 = I_0 = 1$, so that $\mathbb{V}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = I_2 = 1$. Now,

$$\begin{aligned}\Phi_X(u) := \mathbb{E}\left[\mathrm{e}^{uX}\right] = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\mathrm{e}^{ux}\exp\left\{-\frac{x^2}{2}\right\}\mathrm{d}x &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\exp\left\{-\frac{1}{2}\left(x^2-2ux\right)\right\}\mathrm{d}x\\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\exp\left\{-\frac{(x-u)^2}{2}+\frac{u^2}{2}\right\}\mathrm{d}x\\ &= \frac{\mathrm{e}^{u^2/2}}{\sqrt{2\pi}}\int_{\mathbb{R}}\exp\left\{-\frac{y^2}{2}\right\}\mathrm{d}y = \mathrm{e}^{u^2/2}, \qquad \textit{for all } u\in\mathbb{R}.\end{aligned}$$

(ii) Let $Y = \exp\{X\}$. Then

$$\mathbb{E}[Y] = \mathbb{E}\left[\mathrm{e}^X\right] = \Phi_X(1) = \mathrm{e}^{1/2}, \qquad \textit{and} \qquad \mathbb{E}\left[Y^2\right] = \mathbb{E}\left[\mathrm{e}^{2X}\right] = \Phi_X(2) = \mathrm{e}^2,$$

and hence $\mathbb{V}[X] = \Phi_X(2) - \Phi_X(1)^2$. Furthermore, the moment generating function reads

$$\Phi_Y(u) := \mathbb{E}\left[\mathrm{e}^{uY}\right] = \mathbb{E}\left[\mathrm{e}^{u\mathrm{e}^X}\right] = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\mathrm{e}^{u\mathrm{e}^x}\exp\left\{-\frac{x^2}{2}\right\}\mathrm{d}x.$$

Now, for $u>0$, the integrand diverges at positive infinity, and hence the moment generating function is not well defined on the positive half line. Obviously $\Phi_Y(0) = 1$. On the negative half line, Φ_Y is well defined, but no closed form is available.

(iii) The density of Y can be written as, for any $y>0$,

$$f_Y(y) = \partial_y\mathbb{P}(Y\le y) = \partial_y\mathbb{P}(X\le\log(y)) = \frac{1}{y}f_X(\log(y)) = \frac{1}{y\sqrt{2\pi}}\exp\left\{-\frac{\log(y)^2}{2}\right\}.$$