STATISTICAL METHODS IN FINANCE, ASSIGNMENT 1

JACK JACQUIER
MSC MATHEMATICS AND FINANCE, IMPERIAL COLLEGE LONDON

Exercise 1 (Skewness and kurtosis)**.** For a given random variable X on the real line, with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$, recall (see the lecture notes) that the skewness $\mathcal{S}$ and kurtosis κ are defined as

$$\mathcal{S} := \mathbb{E}\left[\left(\frac{X-\mu}{\sigma}\right)^3\right] \qquad \text{and} \qquad \kappa := \mathbb{E}\left[\left(\frac{X-\mu}{\sigma}\right)^4\right].$$

The skewness describes the asymmetry of the distribution, whereas the kurtosis is a measure of its fatness. We usually speak of the excess kurtosis, though, defined as $\kappa_+ := \kappa - 3$.

(i) Using Hölder's inequality in the notes, prove Lyapunov's inequality, namely that

$$\mathbb{E}\left[|X|^p\right] \leq \mathbb{E}\left[|X|^q\right]^{p/q},$$

for all $0 < p < q$ for which both sides of the inequality are finite. Deduce that $\kappa_+ \geq -2$.

(ii) Compute $\mathcal{S}$ and κ for $\mathcal{N}(\mu, \sigma^2)$.

(iii) Compute $\mathcal{S}$ and κ for the Uniform random variable on $[a, b]$.

(iv) Compute $\mathcal{S}$ and κ for the Exponential random variable with intensity $\lambda > 0$, and density

$$f(x) = \partial_x \mathbb{P}[X \leq x] = \lambda \mathrm{e}^{-\lambda x}, \qquad \text{for } x \geq 0.$$

(v) A distribution with $\kappa_+ = 0$ is called mesokurtic. One with $\kappa_+ > 0$ is leptokurtic, and platykurtic if $\kappa_+ < 0$.
- Check the origin and meaning of the Greek words *meso, lepto, platy, kurtos*.
- Let X represent daily logarithmic returns of some data. Using the IPython notebooks, find leptokurtic, platykurtic, or close to mesokurtic examples, and for which $\mathcal{S} > 0$ and $\mathcal{S} < 0$.

Exercise 2 (Convergence and Central Limit Theorem)**.** Consider an iid sequence $(X_i)_{i=1,\ldots,n}$ with common law a Poisson distribution with parameter $\lambda > 0$, that is such that

$$\mathbb{P}[X_1 = k] = \frac{\lambda^k \mathrm{e}^{-\lambda}}{k!}, \qquad \text{for } k = 0, 1, 2, \ldots$$

(i) Compute $\mathbb{E}[X_1]$ and $\mathbb{V}[X_1]$.

(ii) Show that the the empirical average S_n converges in probability to λ as n tends to infinity, where

$$S_n := \frac{1}{n}\sum_{i=1}^{n} X_i, \qquad \text{for } n \geq 1.$$

(iii) Define $T_n := \exp\{-M_n\}$, and show that $(T_n)_{n\geq 1}$ converges in probability to $\mathrm{e}^{-\lambda}$ as n tends to infinity.

(iv) Using the Central Limit Theorem, determine the limiting distribution of $(T_n)_{n\geq 1}$.

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Exercise 3 (Convergence of random variables)**.** We consider a sequence $(X_n)_n$ of random variables on $\mathbb{R}$.

(i) Recall the Borel-Cantelli lemma: *for a sequence $(A_n)_{n\geq 1}$ of events in some given probability space, if $\sum_{n\geq 1}\mathbb{P}(A_n)$ is finite, then $\mathbb{P}(\limsup_{n\uparrow\infty} A_n) = 0$, e.g. the probability that infinitely many events occur is null. Here, the* lim sup *is defined for sequences of events as*

$$\limsup_{n\uparrow\infty} A_n := \cap_{n\geq 1} \cup_{p\geq n} A_k,$$

Consider the case where $X_n = 1$ with probability $1/n$ and zero otherwise. Using the Borel-Cantelli lemma, show that the sequence $(X_n)_n$ converges in probability but not almost surely.

(ii) Show that convergence in probability implies convergence in distribution. You may want to prove first that for any one-dimensional random variables X and Y and any $x \in \mathbb{R}$, $\varepsilon > 0$, we have

$$\mathbb{P}(Y \leq x) \leq \mathbb{P}(X \leq x + \varepsilon) + \mathbb{P}(|Y - X| > \varepsilon).$$

(iii) Show that converges in distribution to a constant implies convergence in probability holds. You may want to use the following result: *the sequence (X_n) converges in distribution to X if and only if*

$$\limsup_{n\uparrow\infty} \mathbb{P}(X_n \in C) \leq \mathbb{P}(X \in C) \quad \textit{holds for any closed set } C.$$

Exercise 4 (Joint distributions)**.** The two questions below are independent.

(i) Let X_1 and X_2 two independent $\mathcal{N}(0,1)$ random variables on $\mathbb{R}$, and, for some $\rho \in [-1,1]$, $\mu_1, \mu_2 \in \mathbb{R}$ and $\sigma_2, \sigma_2 > 0$, define

$$Y_1 := \mu_1 + \sigma_1\left(\rho X_2 + \sqrt{1-\rho^2}X_1\right) \qquad \text{and} \qquad Y_2 := \mu_2 + \sigma_2 X_2.$$

Determine the joint distribution of (Y_1, Y_2), the marginal distribution of Y_1 and Y_2 as well as the conditional distributions $Y_1|Y_2$ and $Y_2|Y_1$. What is the correlation between Y_1 and Y_2?

(ii) Let U_1 and U_2 denote two independent random variables with Uniform distributions on $[0,1]$, and define

$$X_1 := \sqrt{-2\log(U_1)}\ \cos(2\pi U_2),$$
$$X_2 := \sqrt{-2\log(U_1)}\ \sin(2\pi U_2).$$

Determine the joint distribution of (X_1, X_2).

Exercise 5 (Log-normal distribution)**.** The two questions below are independent. Consider the standard Gaussian distribution $X \sim \mathcal{N}(0,1)$, with density

$$f(x) = \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{x^2}{2}\right\}, \qquad \text{for all } x \in \mathbb{R}.$$

(i) Compute $\mathbb{E}[X]$, $\mathbb{V}[X]$ and $\mathbb{E}\left[e^{uX}\right]$ for all $u \in \mathbb{R}$ such that the expectation is well defined.

(ii) Define $Y := \exp\{X\}$. Compute its density, expectation, variance and moment generating function.

(iii) Does Y have a symmetric distribution? Compute its skewness to confirm your guess.