PROBLEM CLASS: FINITE DIFFERENCES FOR AMERICAN OPTIONS

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$ and a $\mathbb{P}$-Brownian motion $(W_t)_{t\in[0,T]}$. Let $(S_t)_{t\in[0,T]}$ be an asset price process $S_t : [0,T] \times \Omega \to \mathbb{R}_+$. Denote by r a risk-free interest rate and consider an equivalent martingale measure $\mathbb{Q} \sim \mathbb{P}$ such that discounted price process is a $\mathbb{Q}$-martingale. Under $\mathbb{Q}$ the dynamics of the stock price read

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma S_t \mathrm{d}W_t^{\mathbb{Q}},$$

where $W^{\mathbb{Q}}$ is a $\mathbb{Q}$-Brownian motion. Consider a European option on S with maturity T and pay-off $g(\cdot)$, and denote the t-time value of the option by $v(t,s)$. We know that $v(t,s)$ is the solution to the well-known Black-Scholes PDE

$$\mathcal{L}^r v(t,s) := \mathcal{L}v(t,s) - rv(t,s) = 0,$$

for all $(t,s) \in \mathbb{R}_+ \times [0,T)$ with terminal condition $v(T,s) = g(s)$, where the differential operator $\mathcal{L}$ is defined as

$$\mathcal{L}v(t,s) := rs\partial_s v(t,s) + \frac{\sigma^2}{2}s^2\partial_{ss}v(t,s) + \partial_t v(t,s).$$

Now, consider an American option on S, the pay-off $A_T = g(s)$ where A_t is a random variable representing the pay-off of an American claim at time t. Since an American option can be exercised at any time before the final maturity, the Snell envelope of the discounted pay-off $v(t,s)$ satisfies

$$v(t,s) := \sup_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}^{\mathbb{Q}}\left\{\mathrm{e}^{-r(T-\tau)}A_\tau | \mathcal{F}_t\right\},$$

where $\mathcal{T}_{t,T}$ is a set of stopping times valued in $[0,T]$. In fact $v(t,s)$ is the value of an American option. The following lemma links the value of an American option to the Black-Scholes PDE via the variational inequality.

Lemma 0.1. *The price of an American option with pay-off $g(\cdot)$ is a function $v : [0,T]\times\mathbb{R}_+ \to \mathbb{R}_+$ satisfying the following variational inequality*

$$\min\{-\mathcal{L}^r v(t,s), v(t,s) - g(s)\} = 0, \qquad \textit{for all } (t,s) \in [0,T) \times \mathbb{R}_+.$$

In particular the lemma implies that $\mathcal{L}^r v(t,s) = 0$ for all (t,s) in the continuation region $\mathcal{C}$ and $v(t,s) = g(s)$ for all (t,s) in the exercise region $\mathcal{S}$, where

$$\mathcal{C} := \{(t,s) \in [0,T) \times \mathbb{R}_+ : v(t,s) > g(s)\},$$

$$\mathcal{S} := \{(t,s) \in [0,T] \times \mathbb{R}_+ : v(t,s) = g(s)\}.$$

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Lemma 0.2. *Suppose that there exists a function* $w(t,s) \in C^{1,2}([0,T] \times \mathbb{R}_+)$ *satisfying*

$$\begin{aligned} \mathcal{L}^r w(t,s) &\leq 0, \\ g(s) &\leq w(t,s). \end{aligned} \tag{1}$$

Then $v(t,s) \leq w(t,s)$ *for all* $(t,s) \in [0,T) \times \mathbb{R}_+$.

The second lemma in particular implies that the price of the European call option on a non-dividend paying stock is equal to the price of an American option with the same characteristics. The usual Put-Call parity for European Options does not hold in general. However one can find a similar relationship for American Options.

Lemma 0.3. *Let* P *be the price of an American Put Option and* C *the price of an American Call Option with strike* K *and maturity* T. *Let the price of the underlying today is* S_0 *and* r *be the risk-free interest rate. The the following inequalities hold*

$$S_0 - K \leq C - P \leq S_0 - K e^{-rT}.$$

Proof. Exercise. □