CASE STUDY: VARIATIONS AROUND CRANK-NICOLSON
M5MF2 NUMERICAL METHODS IN FINANCE, IMPERIAL COLLEGE LONDON

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We would like to understand the behaviour of the Crank-Nicolson scheme for Call option pricing in the Black-Scholes framework. Following the notations in the notes, consider the Black-Scholes equation (not the heat equation) for a Call option with strike $K > 0$, maturity $T > 0$, written on a stock $(S_t)_{t\geq 0}$. We assume no interest rate and no dividend. For the scheme, we consider a uniform grid $\mathcal{I} \times \mathcal{J}$ approximating $[0, T] \times [\underline{S}, \overline{S}]$, where $\mathcal{I} := \{0, 1, \ldots, n\}$ and $\mathcal{J} := \{0, 1, \ldots, m\}$ and $0 < \underline{S} < \overline{S}$. A point (i, j) on the grid corresponds to $(i\delta_T, \underline{S} + j\delta_S)$, where $\delta_T := \frac{T}{n}$ and $\delta_S := \frac{\overline{S} - \underline{S}}{m}$. We shall denote by $u(\cdot)$ the value function, and by $u_{i,j}$ the solution to the finite difference scheme at the point (i, j) on the grid. We shall consider the following values for the parameters:

$$S_0 = 100, \qquad K \in \{80, 100, 120\}, \qquad T = 1, \qquad \sigma = 20\%.$$

In all questions below, discuss the three possible strike values and explain if any differences arise.

(1) Write the Crank-Nicolson scheme for the Black-Scholes PDE, with boundary conditions:

$$u_{n,j} = (\underline{S} + j\delta_S - K)_+\,, \text{for } j \in \mathcal{J}, \qquad \begin{cases} u_{i,0} = & 0, \\ u_{i,m} = & (\underline{S} + m\delta_S - K)_+\,, \end{cases} \qquad \text{for } i \in \mathcal{I} \setminus \{n\}.$$

(2) For $n \in \{50, 200, 1000\}$, discuss the convergence and speed of the algorithm to the true solution as m increases.

(3) Assume now that, instead of Dirichlet boundary conditions at the space boundaries, one considers Neumann conditions of the form

$$\partial_{SS} u(t, \overline{S}) = \partial_{SS} u(t, \underline{S}) = 0.$$

Implement the Crank-Nicolson scheme in that case, making sure that the error magnitude remains the same at the boundary (hint: one may assume that the new boundary condition also holds at the first interior point of the domain), and compare the convergence/errors with the Dirichlet-type scheme above.

(4) Following the example in the notes, modify the Crank-Nicolson scheme above with a non-uniform grid (centered around the strike), and discuss.

Date: January 27, 2017.