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Numerical analysis for Spread option pricing model of markets with finite liquidity: first-order feedback model

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In this paper, we discuss the numerical analysis and the pricing and hedging of European Spread options on correlated assets when, in contrast to the standard framework and consistent with a market with imperfect liquidity, the option trader's trading in the stock market has a direct impact on one of the stocks price. We consider a *first-order feedback model* which leads to a linear partial differential equation. The Peaceman-Rachford scheme is applied as an alternating direction implicit method to solve the equation numerically. We also discuss the stability and convergence of this numerical scheme. Finally, we provide a numerical analysis of the effect of the illiquidity in the underlying asset market on the replication of an European Spread option; compared to the Black–Scholes case, a trader generally buys less stock to replicate a call option.

Keywords: Spread option pricing; price impact; illiquid markets; Peaceman-Rachford scheme

2010 AMS Subject Classifications: 91G20; 35K15; 65M06

1. Introduction

Black and Scholes [2] and most of the work undertaken in mathematical finance assume that the market in the underlying asset is infinitely (or perfectly) liquid, such that trading had no effect on the price of underlying asset. In the market with finite liquidity, trading does affect the underlying asset price, regardless of her trading size. The model we consider involves the price impact due to the action of a large trade that may itself impact the price, independent of all the other factors affecting the price dynamics; this is termed price impact. In the presence of such a price impact, the most important issue is how the impact price can affect the replication of an option. This encouraged researchers to develop the Black–Scholes model to models that involve the price impact due to a large trader who is able to move the price by his/her actions. An excellent survey of these research can be found in [12,16,25,41].

In [37], we investigated the effects of the full-feedback model in which price impact is fully incorporated into the model and results in highly nonlinear partial differential equation. Our purpose of this paper is to investigate the effects of imperfect liquidity on the replication of an European Spread option by a typical option trader, when the hedging strategy does not take into account the feedback effect (we term first-order feedback model). We assume that a Spread

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option is to be hedged and furthermore that the hedger holds the number of stocks dictated by the analytical Black–Scholes delta, rather than the delta from the modified option price. This leads to the linear partial differential equation (PDE), which is somewhat easier to solve than the full-feedback model but still has important and interesting differences from the classical Black–Scholes PDE.

Spread option is the simplest example of multi-assets derivative, whose payoff is the difference between the prices of two or more assets; for instance let the prices of two underlying assets at time $t \in [0, T]$ be $S_1(t)$ and $S_2(t)$, then the payoff function of a European Spread option with maturity T is $[S_1(T) - S_2(T) - k]^+$ (here k is the strike of the option and the function x^+ is defined as $x^+ = \max(x, 0)$). Therefore the holder of a European Spread option has the right but not the obligation to buy the spread $S_1(T) - S_2(T)$ at the prespecified price k and maturity T. In general, there is no analytical formula for the price of multi-assets options. The only exception is Margrabe formula for exchange options (Spread options with a strike of zero) [26]. Kirk [23] found an analytical approximation for Spread options with k positive and close to zero.

Several Spread options are traded in the markets, e.g. fixed income Spread options, foreign exchange and commodity Spread options. In this work, we focus on commodity Spread options. Spread options, in commodity market, hedge the risk of price fluctuations between input and output products. In order to price them one needs to take into account the characteristics of the commodities prices they are written upon. During the past decades, several stochastic models for commodity prices have been introduced. The first models assumed that the price processes follow a geometric Brownian motion and that all the uncertainty could be summarized by one factor. Models of this type include Cox and Schwartz [6] for pricing commodity-linked securities, Brennan and Schwartz [3], Paddock et al. [30], and Cortazar and Schwartz [4] for valuing real assets. Mean reverting price processes were considered by Schwartz [35]. Most models assumed that there is a single source of randomness driving the prices of the commodities. Since empirical evidence suggests more sources of randomness, several two- and three-factor models were subsequently developed. In their two-factor model, Gibson and Schwartz [13] assumed that the spot price of the commodity and the convenience yield (the difference between the interest rate and the cost of carry) follow a joint stochastic process. Cortazar and Schwartz in [5] took a different approach; they used all the information contained in the term structure of commodity futures prices together with the historical volatilities of future return for different maturities. A good comparison among these models was performed in [35]. Schwartz and Smith [36] modelled the log spot price as the sum of two stochastic factors and they showed that this model is equivalent to the Gibson and Schwartz [13] model. Pascheke and Prokopczuk [31] developed a continuous time factor model which allows for higher-order autoregressive and moving average components. A review of these models is done in [1]. There are several types of commodity Spread options, some of the popular ones are:

Crush Spread option. In the agricultural markets, the Chicago Board of Trade the so-called crush spread which exchanges soyabeans (as a unrefined product) with a combination of soyabean oil and soyabean meal (as the derivative products). Johnson *et al.* [22] studied Spread options in the agricultural markets.

Spark Spread option. In the energy markets, spark Spread options are a spread between natural gas and power (electricity). Girma and Paulson [14,15] studied these type of options.

Crack Spread options. A Crack Spread represents the differential between the price of crude oil and petroleum products (gasoline or heating oil). The underlying indexes comprise futures prices of crude oil, heating oil and unleaded gasoline. Details of Crack Spread options can be found in the New York Mercantile Exchange Crack Spread Handbook [29]. Our paper is aimed at pricing Crack Spread options. In the oil markets with finite liquidity, trading does affect the underlying assets price. In our study, we are going to investigate the effects of price impact when trading affects only the crude oil price and not the petroleum products. Our model is

related to the constant convenience yield model of [35]. In their model of the commodity price, the rate of return is affected by a stochastic convenience yield; in our model, due to the liquidity risk, both the rate of return and the volatility of the risky asset are affected by stochastic factors.

We study in this work a splitting scheme of the alternating direction implicit (ADI) type associated with a two-dimensional PDE (which characterizes the option price). This method has the desirable stability features of the Crank Nicolson method, but it proceeds in two steps. The first half step is taken implicitly in one space variable and explicitly in the other, while the second half step reverses the explicit and implicit variables. Thus the numerical problem reduces to solving two matrix equations. ADI method goes back to [32] and has been further developed in many works, e.g. [18,19,42] (for financial applications see [7,24,27,28,34]). ADI schemes were not originally developed for multi-dimensional convection-diffusion equations with mixed derivative terms. The problems generated by the cross-derivatives were first discussed in [39,40]. Furthermore, Pospisil and Vecer [33] applied the Peaceman-Rachford and Douglas-Rachford schemes as ADI method; Hout and Foulon [20] investigated four splitting schemes of ADI type: the Douglas scheme, the Craig-Sneyd (CS) scheme, the modified CS scheme and the Hundsdorfer-Verwer scheme; Haentjens [17] investigated the effectiveness of ADI time discretization schemes in the numerical solution of three-dimensional Heston-Hull-White PDE; Dang et al. [8] employed the ADI method based on Hundsdorfer and Verwer (HV) splitting approach for pricing foreign exchange interest rate hybrid derivatives. In this work, we use the Peaceman and Rachford scheme that was introduced first in [32]. Since the resulting multi-dimensional linear PDE has mixed derivative terms we have to adjust the Peaceman and Rachford scheme.

This paper is organized as follows: in Section 2, we introduce our problem and discuss the general framework we use. In Section 3 we propose the splitting scheme of the ADI type (subsequently, we discuss the stability and the convergence of the scheme). In Section 4, we carry out several numerical experiments and provide a numerical analysis. Section 5 contains the concluding remarks.

2. The model setup

In this section we describe the setup for Spread option pricing. Our model of a financial market, based on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, \mathbf{P})$ that satisfies the usual conditions, consists of two assets. Their prices are modelled by a two-dimensional Ito-process $S(t) = (S_1(t), S_2(t))$. All the stochastic processes in this work are assumed to be $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted. Their dynamics are given by the following stochastic differential equations, in which $W(t) = (w_1(t), w_2(t))$ is defined a two-dimensional standard Brownian motion with $\{\mathcal{F}_t\}_{t \in [0,T]}$ being its natural filtration augment by all **P**-null sets:

$$\frac{\mathrm{d}S_i(t)}{S_i(t)} = \mu_i(t, S_i(t)) \,\mathrm{d}t + \sigma_i(t, S_i(t)) \,\mathrm{d}w_i(t); \quad i = 1, 2, \tag{1}$$

where w_1 and w_2 are two correlated Brownian motions with correlation ρ , $\mu_i(t, S_i(t))$ and $\sigma_i(t, S_i(t))$ are the expected return and the volatility of stock *i* in the absence of price impact. It is possible to add a partial price impact for the first stock, i.e.

$$dS_1(t) = \mu_1(t, S_1(t))S_1(t) dt + \sigma_1(t, S_1(t))S_1(t) dw_1(t) + \lambda(t, S_1) df(t, S_1, S_2), dS_2(t) = \mu_2(t, S_2(t))S_2(t) dt + \sigma_2(t, S_2(t))S_2(t) dw_2(t),$$
(2)

where $\lambda(t, S_1) \ge 0$ is an arbitrary function and $\lambda(t, S_1) df(t, S_1, S_2)$ represents the price impact of the investor's trading. We see that the two-dimensional classical Black–Scholes model is a special case of this model with $\lambda(t, S_1(t)) = 0$.

Our aim is to price a Spread option under the modified stochastic process (2), with the following payoff at maturity T (a call at this case):

$$h(S_1(T), S_2(T)) = (S_1(T) - S_2(T) - k)^+,$$
(3)

where k is the strike price. In order to provide a derivation of the pricing PDE considered in this work, we use the well-known generalized Black–Scholes equation (more details in [9]). This leads to the following pricing PDE for the modified stochastic process incorporating the forcing term (2)

$$\begin{aligned} \frac{\partial V}{\partial t}(t, S_1, S_2) &+ \frac{1}{2(1 - \lambda(t, S_1)(\partial f / \partial S_1)(t, S_1, S_2))^2} \left(\sigma_1^2 S_1^2 + \lambda^2(t, S_1) \sigma_2^2 S_2^2 \left(\frac{\partial f}{\partial S_2}(t, S_1, S_2) \right)^2 \\ &+ 2\rho \sigma_1 \sigma_2 S_1 S_2 \lambda(t, S_1) \frac{\partial f}{\partial S_2}(t, S_1, S_2) \right) \frac{\partial^2 V}{\partial S_1^2}(t, S_1, S_2) + \frac{1}{2} \sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2}(t, S_1, S_2) \\ &+ \frac{1}{(1 - \lambda(t, S_1)(\partial f / \partial S_1)(t, S_1, S_2))} \left(\sigma_1 \sigma_2 \rho S_1 S_2 + \lambda(t, S_1) \sigma_2^2 S_2^2 \frac{\partial f}{\partial S_2}(t, S_1, S_2) \right) \\ &\times \frac{\partial^2 V}{\partial S_1 \partial S_2}(t, S_1, S_2) + r \left(S_1 \frac{\partial V}{\partial S_1}(t, S_1, S_2) + S_2 \frac{\partial V}{\partial S_2}(t, S_1, S_2) \right) \\ &- r V(t, S_1, S_2) = 0, \quad 0 < S_1, S_2 < \infty, \ 0 \le t < T. \end{aligned}$$

Here *r* is the riskless rate of the money market. Consistent with standard Black–Scholes arguments, the drift of the modified process $\mu(t, S(t))$ does not appear in the option pricing PDE. In the context of markets with finite liquidity, we can define $f(t, S_1, S_2)$ to be the number of extra shares traded due to some deterministic hedging strategy, and $\lambda(t, S_1(t))$ as some function dependent on how we choose to model the form of price impact. Here, similar to [25], we consider $\lambda(t, S_1(t)) = \varepsilon \hat{\lambda}(t, S_1(t))$, with $\hat{\lambda}(t, S_1)$ a function such that $\hat{\lambda}(T, S_1) = 0$ and $\varepsilon > 0$ the constant price impact coefficient. In the first-order feedback model $f(t, S_1, S_2)$ in Equation (4) is

$$f(t, S_1, S_2) = \frac{\partial V^{\mathrm{BS}}}{\partial S_1}(t, S_1, S_2),$$

where $V^{BS}(t, S_1, S_2)$ is the Black–Scholes value (see [16]). This leads to the following linear PDE:

$$\begin{aligned} \frac{\partial V}{\partial t}(t, S_1, S_2) &+ \frac{1}{2(1 - \lambda(t, S_1)(\partial^2 V^{BS}/\partial S_1^2)(t, S_1, S_2))^2} \\ &\times \left(\sigma_1^2 S_1^2 + \lambda^2(t, S_1)\sigma_2^2 S_2^2 \left(\frac{\partial^2 V^{BS}}{\partial S_1 \partial S_2}(t, S_1, S_2)\right)^2 + 2\rho\sigma_1\sigma_2 S_1 S_2 \lambda(t, S_1) \frac{\partial^2 V^{BS}}{\partial S_1 \partial S_2}(t, S_1, S_2)\right) \\ &\times \frac{\partial^2 V}{\partial S_1^2}(t, S_1, S_2) + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2}(t, S_1, S_2) + \frac{1}{(1 - \lambda(t, S_1)(\partial^2 V^{BS}/\partial S_1^2)(t, S_1, S_2))}) \\ &\times \left(\sigma_1\sigma_2\rho S_1 S_2 + \lambda(t, S_1)\sigma_2^2 S_2^2 \frac{\partial^2 V^{BS}}{\partial S_1 \partial S_2}(t, S_1, S_2)\right) \frac{\partial^2 V}{\partial S_1 \partial S_2}(t, S_1, S_2) + r \\ &\times \left(S_1 \frac{\partial V}{\partial S_1}(t, S_1, S_2) + S_2 \frac{\partial V}{\partial S_2}(t, S_1, S_2)\right) \\ &- rV(t, S_1, S_2) = 0, \quad 0 < S_1, S_2 < \infty, \quad 0 \le t < T. \end{aligned}$$
(5)

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For investigating the treatment of boundary conditions we apply Fichera's theory [11]. In order to determine the subsets where boundary conditions can be imposed, we need to evaluate the Fichera function. Equation (5) is defined on D where

$$D = \{(t, S_1, S_2) | 0 < t \le T, 0 < S_1 < \infty, 0 < S_2 < \infty\}$$

The corresponding coefficient matrix is

$$A = \begin{pmatrix} a_{11} & a_{12} & 0\\ a_{21} & a_{22} & 0\\ 0 & 0 & 0 \end{pmatrix}.$$

where the components of *A* are the following:

$$a_{11} = \frac{1}{2(1 - \lambda V_{S_1S_1}^{BS})^2} (\sigma_1^2 S_1^2 + \lambda^2 \sigma_2^2 S_2^2 (V_{S_1S_2}^{BS})^2 + 2\rho \sigma_1 \sigma_2 S_1 S_2 \lambda V_{S_1S_2}^{BS}),$$

$$a_{12} = \frac{1}{2(1 - \lambda V_{S_1S_1}^{BS})} (\sigma_1 \sigma_2 \rho S_1 S_2 + \lambda \sigma_2^2 S_2^2 V_{S_1S_2}^{BS}),$$

$$a_{21} = \frac{1}{2(1 - \lambda V_{S_1S_1}^{BS})} (\sigma_1 \sigma_2 \rho S_1 S_2 + \lambda \sigma_2^2 S_2^2 V_{S_1S_2}^{BS}),$$

$$a_{22} = \frac{1}{2} \sigma_2^2 S_2^2.$$
(6)

A is a singular matrix everywhere. For the boundaries $S_1 = 0$, $S_2 = 0$ and t = 0 we have the corresponding inward normals n = (1, 0, 0), (0, 1, 0) and (0, 0, 1), and the inward normal on t = T is (0, 0, -1). We let \sum^0 be the subset of ∂D where $\langle An, n \rangle = 0$. We observe that $\langle An, n \rangle = 0$ at all of the boundary points so $\partial D = \sum^0$. The Fichera function is

$$h = \left[rS_{1} + \frac{\lambda V_{S_{1}S_{1}S_{1}}^{BS}}{2(1 - \lambda V_{S_{1}S_{1}}^{BS})^{4}} (\sigma_{1}^{2}S_{1}^{2} + \lambda^{2}\sigma_{2}^{2}S_{2}^{2}(V_{S_{1}S_{2}}^{BS})^{2} + 2\rho\sigma_{1}\sigma_{2}S_{1}S_{2}\lambda V_{S_{1}S_{2}}^{BS}) \right]$$

$$- \frac{1}{2(1 - \lambda V_{S_{1}S_{1}}^{BS})^{2}} (2\sigma_{1}^{2}S_{1} + 2\lambda^{2}\sigma_{2}^{2}S_{2}^{2}V_{S_{1}S_{2}}^{BS} + 2\rho\sigma_{1}\sigma_{2}S_{2}\lambda V_{S_{1}S_{2}}^{BS} + 2\rho\sigma_{1}\sigma_{2}S_{1}S_{2}\lambda V_{S_{1}S_{2}}^{BS}) - \frac{1}{2(1 - \lambda V_{S_{1}S_{1}}^{BS})} (\rho\sigma_{1}\sigma_{2}S_{1} + 2\lambda\sigma_{2}^{2}S_{2}V_{S_{1}S_{2}}^{BS} + \lambda\sigma_{2}^{2}S_{2}^{2}V_{S_{1}S_{2}}^{BS}) + \frac{\lambda V_{S_{1}S_{1}}^{BS}}{(1 - \lambda V_{S_{1}S_{1}}^{BS})^{2}} (\sigma_{1}\sigma_{2}\rho S_{1}S_{2} + \lambda\sigma_{2}^{2}S_{2}^{2}V_{S_{1}S_{2}}^{BS}) \right] n_{1}$$

$$+ \left[rS_{2} - \sigma_{2}^{2}S_{2} - \frac{1}{2(1 - \lambda V_{S_{1}S_{1}}^{BS})} (\rho\sigma_{1}\sigma_{2}S_{2} + \lambda\sigma_{2}^{2}S_{2}^{2}V_{S_{1}S_{2}}^{BS}) + \frac{\lambda V_{S_{1}S_{1}}^{BS}}{(1 - \lambda V_{S_{1}S_{1}}^{BS})^{2}} (\sigma_{1}\sigma_{2}\rho S_{1}S_{2} + \lambda\sigma_{2}^{2}S_{2}^{2}V_{S_{1}S_{2}}^{BS}) \right] n_{2} - n_{3}.$$

$$(7)$$

On $S_2 = 0$ we see that $h(S_1, 0, t) = 0$ and according to Fichera's theory no boundary data should be given. Instead the differential equation should hold on $S_2 = 0$. On $S_1 = 0$ we see that $h(0, S_2, t) = 0$ and according to Fichera's theory no boundary data should be given. Instead the differential equation should hold on $S_1 = 0$. On t = 0 we see that $h(S_1, v, 0) = -1$ so we can impose the payoff of the option at maturity as initial condition on Equation (5). At t = T the differential equation holds.

Remark 2.1 In [37], we have investigated a full-feedback model, where the price impact is fully incorporated into the model. The corresponding equation is

$$\begin{aligned} \frac{\partial V}{\partial t}(t, S_1, S_2) &+ \frac{1}{2(1 - \lambda(t, S_1)(\partial^2 V / \partial S_1^2)(t, S_1, S_2))^2} \\ &\times \left(\sigma_1^2 S_1^2 + \lambda^2(t, S_1)\sigma_2^2 S_2^2 \left(\frac{\partial^2 V}{\partial S_1 \partial S_2}(t, S_1, S_2)\right)^2 + 2\rho\sigma_1\sigma_2 S_1 S_2 \lambda(t, S_1) \frac{\partial^2 V}{\partial S_1 \partial S_2}(t, S_1, S_2)\right) \\ &\times \frac{\partial^2 V}{\partial S_1^2}(t, S_1, S_2) + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2}(t, S_1, S_2) + \frac{1}{(1 - \lambda(t, S_1)(\partial^2 V / \partial S_1^2)(t, S_1, S_2))} \\ &\times \left(\sigma_1\sigma_2\rho S_1 S_2 + \lambda(t, S_1)\sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_1 \partial S_2}(t, S_1, S_2)\right) \frac{\partial^2 V}{\partial S_1 \partial S_2}(t, S_1, S_2) \\ &+ r \left(S_1 \frac{\partial V}{\partial S_1}(t, S_1, S_2) + S_2 \frac{\partial V}{\partial S_2}(t, S_1, S_2)\right) - rV(t, S_1, S_2) = 0, \ 0 < S_1, S_2 < \infty, \ 0 \le t < T, \\ V(T, S_1, S_2) = h(S_1, S_2), \ 0 < S_1, S_2 < \infty. \end{aligned}$$

$$(8)$$

The first-order approximation is

$$V(t, S_1, S_2) = V^0(t, S_1, S_2) + \varepsilon V^1(t, S_1, S_2) + o(\varepsilon^2),$$

where $V^0(t, S_1, S_2)$ is the Black–Scholes price for European Spread option, i.e.

$$\frac{\partial V^{0}}{\partial t} + \frac{\sigma_{1}^{2} S_{1}^{2}}{2} \frac{\partial^{2} V^{0}}{\partial S_{1}^{2}} + \frac{\sigma_{2}^{2} S_{2}^{2}}{2} \frac{\partial^{2} V^{0}}{\partial S_{2}^{2}} + \sigma_{1} \sigma_{2} S_{1} S_{2} \rho \frac{\partial^{2} V^{0}}{\partial S_{1} \partial S_{2}} + r \left[S_{1} \frac{\partial V^{0}}{\partial S_{1}} + S_{2} \frac{\partial V^{0}}{\partial S_{2}} \right] - r V^{0} = 0,$$

$$V^{0}(T, S_{1}, S_{2}) = \max(S_{1}(T) - S_{2}(T) - k, 0), \quad 0 < S_{1}, S_{2} < \infty,$$
(9)

and $V^1(t, S_1, S_2)$ is the solution of the following problem

$$\frac{\partial V^{1}}{\partial t} + \frac{\sigma_{1}^{2} S_{1}^{2}}{2} \frac{\partial^{2} V^{1}}{\partial S_{1}^{2}} + \frac{\sigma_{2}^{2} S_{2}^{2}}{2} \frac{\partial^{2} V^{1}}{\partial S_{2}^{2}} + \sigma_{1} \sigma_{2} S_{1} S_{2} \rho \frac{\partial^{2} V^{1}}{\partial S_{1} \partial S_{2}} + r \left[S_{1} \frac{\partial V^{1}}{\partial S_{1}} + S_{2} \frac{\partial V^{1}}{\partial S_{2}} \right] - r V^{1} = G,$$

$$V^{1}(T, S_{1}, S_{2}) = 0, \quad 0 < S_{1}, S_{2} < \infty.$$
(10)

Here

$$G = -\varepsilon \left(2\rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 V^0}{\partial S_1 \partial S_2} \frac{\partial^2 V^0}{\partial S_1^2} + \sigma_1^2 S_1^2 \left(\frac{\partial^2 V^0}{\partial S_1^2} \right)^2 + \sigma_2^2 S_2^2 \left(\frac{\partial^2 V^0}{\partial S_1 \partial S_2} \right)^2 \right).$$

3. Numerical solution of partial differential equation

3.1 The ADI

In this section, we present a numerical method for solving the pricing partial differential equation

$$\begin{aligned} \frac{\partial V}{\partial t} &+ \frac{1}{2(1 - \lambda(\partial^2 V^{\text{BS}}/\partial x^2))^2} \left(\sigma_1^2 x^2 + \lambda^2 \sigma_2^2 y^2 \left(\frac{\partial^2 V^{\text{BS}}}{\partial x \partial y} \right)^2 + 2\rho \sigma_1 \sigma_2 xy \lambda \frac{\partial^2 V^{\text{BS}}}{\partial x \partial y} \right) \frac{\partial^2 V}{\partial x^2} \\ &+ \frac{1}{2} \sigma_2^2 y^2 \frac{\partial^2 V}{\partial y^2} + \frac{1}{(1 - \lambda(\partial^2 V^{\text{BS}}/\partial x^2))} \left(\sigma_1 \sigma_2 \rho xy + \lambda \sigma_2^2 y^2 \frac{\partial^2 V^{\text{BS}}}{\partial x \partial y} \right) \frac{\partial^2 V}{\partial x \partial y} \end{aligned}$$
(11)
$$&+ r \left(x \frac{\partial V}{\partial x} + y \frac{\partial V}{\partial y} \right) - rV = 0,$$

$$V(T, x, y) = h(x, y), \quad 0 < x, y < \infty,$$

where the functions V := V(t, x, y), $V^{BS} := V^{BS}(t, x, y)$ are defined on $[0, T] \times [0, \infty) \times [0, \infty)$ and $\lambda := \lambda(t, x)$ on $[0, T] \times [0, \infty)$. For the sake of notation, we write the following operators:

$$L = \frac{\partial}{\partial t} + A_x + A_y + A_{xy},\tag{12}$$

where

$$A_{x}V = \frac{1}{2(1 - \lambda(\partial^{2}V^{BS}/\partial x^{2}))^{2}} \left(\sigma_{1}^{2}x^{2} + \lambda^{2}\sigma_{2}^{2}y^{2} \left(\frac{\partial^{2}V^{BS}}{\partial x \partial y} \right)^{2} + 2\rho\sigma_{1}\sigma_{2}xy\lambda \frac{\partial^{2}V^{BS}}{\partial x \partial y} \right) \frac{\partial^{2}V}{\partial x^{2}} + rx \frac{\partial V}{\partial x} - r\Theta,$$

$$A_{y}V = \frac{1}{2}\sigma_{2}^{2}y^{2} \frac{\partial^{2}V}{\partial y^{2}} + ry \frac{\partial V}{\partial y} - r(1 - \Theta),$$

$$A_{xy}V = \frac{1}{(1 - \lambda(\partial^{2}V^{BS}/\partial x^{2}))} \left(\sigma_{1}\sigma_{2}\rho xy + \lambda\sigma_{2}^{2}y^{2} \frac{\partial^{2}V^{BS}}{\partial x \partial y} \right) \frac{\partial^{2}V}{\partial x \partial y},$$
(13)

and $0 \le \Theta \le 1$. While symmetry considerations might speak for an $\Theta = \frac{1}{2}$, it is computationally simpler to use $\Theta = 0$ or $\Theta = 1$, i.e. to include the *rV*- term fully in one of the two operators. Hence, we can write

$$LV = 0, \quad 0 < x, y < \infty, \quad 0 < t < T,$$

$$V(T, x, y) = h(x, y), \quad 0 < x, y < \infty.$$
(14)

In order to define a numerical solution to the equation, we need to truncate the spatial domain to a bounded area as $\{(x, y); 0 \le x \le x_{\max}, 0 \le y \le y_{\max}\}$. We follow [21] in choosing the upper bounds of the domain. The upper bounds should be large enough to include the stock price limits within which there is a price impact. Let us introduce a grid of points in the time interval and in

the truncated spatial domain

$$t_{l} = l\Delta t, \quad l = 0, 1, \dots, L, \ \Delta t = \frac{T}{L},$$

$$x_{m} = m\Delta x, \quad m = 0, 1, \dots, M, \ \Delta x = \frac{x_{\max}}{M},$$

$$y_{n} = n\Delta y, \quad n = 0, 1, \dots, N, \ \Delta y = \frac{y_{\max}}{N}.$$
(15)

For the simplicity of notation, we assume that $x_{max} = y_{max}$ and $\Delta x = \Delta y$. Functions V(t, x, y) and $V^{BS}(t, x, y)$ at a point of the grid will be denoted as $V_{m,n}^l = V(t_l, x_m, y_n)$ and $V_{m,n}^{BS,l} = V^{BS}(t_l, x_m, y_n)$. Furthermore, let us introduce the approximations

$$\frac{\partial V}{\partial x}(t_{l}, x_{m}, y_{n}) = \frac{V_{m+1,n}^{l} - V_{m-1,n}^{l}}{2\Delta x} + O(\Delta^{2}x),$$

$$\frac{\partial V}{\partial y}(t_{l}, x_{m}, y_{n}) = \frac{V_{m,n+1}^{l} - V_{m,n-1}^{l}}{2\Delta y} + O(\Delta^{2}y),$$

$$\frac{\partial^{2} V}{\partial x^{2}}(t_{l}, x_{m}, y_{n}) = \frac{V_{m+1,n}^{l} - 2V_{m,n}^{l} + V_{m-1,n}^{l}}{(\Delta x)^{2}} + O(\Delta^{2}x),$$

$$\frac{\partial^{2} V}{\partial y^{2}}(t_{l}, x_{m}, y_{n}) = \frac{V_{m,n+1}^{l} - 2V_{m,n}^{l} + V_{m,n-1}^{l}}{(\Delta y)^{2}} + O(\Delta^{2}y),$$

$$\frac{\partial^{2} V}{\partial x\partial y}(t_{l}, x_{m}, y_{n}) = \frac{V_{m+1,n+1}^{l} - V_{m-1,n+1}^{l} - V_{m+1,n-1}^{l} + V_{m-1,n-1}^{l}}{4\Delta x\Delta y} + O(\Delta^{2}x + \Delta^{2}y).$$
(16)

Let symbols A_{dx} , A_{dy} and A_{dxdy} denote second-order approximations of the operators A_x , A_y and A_{xy} obtained by using Equation (16) into Equation (13).

We can use ADI because the differential operator can be split as in Equation (13). The general idea is to split a time step in two and to take one operator or one space coordinate at a time (see more details in [10,38]). In this work, particularly we use the Peacman–Rachford scheme.

Taking our inspiration from the Crank–Nicolson method we begin discretizing (11) in the time-direction

$$V_t((l+1/2)\Delta t, x, y) = \frac{V^{l+1} - V^l}{\Delta t} + O(\Delta t^2),$$

$$(A_x + A_y + A_{xy})V = \frac{1}{2}A_x(V^{l+1} + V^l) + \frac{1}{2}A_y(V^{l+1} + V^l) + \frac{1}{2}A_{xy}(V^{l+1} + V^l) + O(\Delta t^2).$$
(17)

Insert in Equation (12), multiply by Δt , and rearrange

$$(I - \frac{1}{2}\Delta tA_x - \frac{1}{2}\Delta tA_y)V^l = (I + \frac{1}{2}\Delta tA_x + \frac{1}{2}\Delta tA_y)V^{l+1} + \frac{1}{2}\Delta tA_{xy}(V^{l+1} + V^l) + O(\Delta t^3),$$
(18)

where *I* denotes the identity operator. If we add $\frac{1}{4}\Delta t^2 A_x A_y V^l$ on the left side and $\frac{1}{4}\Delta t^2 A_x A_y V^{l+1}$ on the right side then we commit an error which is $O(\Delta t^3)$ and therefore can be included in that term

$$(I - \frac{1}{2}\Delta tA_x)(I - \frac{1}{2}\Delta tA_y)V^l = (I + \frac{1}{2}\Delta tA_x)(I + \frac{1}{2}\Delta tA_y)V^{l+1} + \frac{1}{2}\Delta tA_{xy}(V^{l+1} + V^l) + O(\Delta t^3).$$
(19)

Now, we discretize in the space coordinates replacing A_x by A_{dx} , A_y by A_{dy} and A_{xy} by $A_{dx dy}$

$$(I - \frac{1}{2}\Delta t A_{dx})(I - \frac{1}{2}\Delta t A_{dy})V^{l} = (I + \frac{1}{2}\Delta t A_{dx})(I + \frac{1}{2}\Delta t A_{dy})V^{l+1} + \frac{1}{2}\Delta t A_{dx\,dy}(V^{l+1} + V^{l}) + O(\Delta t^{3}) + O(\Delta t \Delta x^{2}),$$
(20)

and this gives rise to the Peaceman-Rachford method

$$\left(I - \frac{\Delta t}{2} A_{dx}\right) V^{l+1/2} = \left(I + \frac{\Delta t}{2} A_{dy}\right) V^{l+1} + \alpha,$$

$$\left(I - \frac{\Delta t}{2} A_{dy}\right) V^{l} = \left(I + \frac{\Delta t}{2} A_{dx}\right) V^{l+1/2} + \beta,$$
(21)

where auxiliary function $V^{l+1/2}$ links above equations. We have introduced the values α and β to take into account the mix derivative term because it is not obvious how this term should be split. In order to correspond the solution (21) by the solution to Equation (20), we have the requirement that

$$\left(I + \frac{\Delta t}{2}A_{dx}\right)\alpha + \left(I - \frac{\Delta t}{2}A_{dx}\right)\beta = \frac{1}{2}\Delta t A_{dx\,dy}(V^{l+1} + V^l),\tag{22}$$

where a discrepancy of order $O(\Delta t^3)$ may be allowed with reference to a similar term in Equation (19). One of the possible choices for α and β is

$$\alpha = \frac{\Delta t}{2} A_{\mathrm{dx}\,\mathrm{dy}} V^{l+1}, \quad \beta = \frac{\Delta t}{2} A_{\mathrm{dx}\,\mathrm{dy}} V^{l+1/2}. \tag{23}$$

Finally, the Peaceman–Rachford scheme for V in Equation (11) is obtained as follows:

$$\left(I - \frac{\Delta t}{2}A_{dx}\right)V^{l+1/2} = \left(I + \frac{\Delta t}{2}A_{dy}\right)V^{l+1} + \frac{\Delta t}{2}A_{dx\,dy}V^{l+1},$$

$$\left(I - \frac{\Delta t}{2}A_{dy}\right)V^{l} = \left(I + \frac{\Delta t}{2}A_{dx}\right)V^{l+1/2} + \frac{\Delta t}{2}A_{dx\,dy}V^{l+1/2}.$$
(24)

In a first step we compute $V^{l+1/2}$ using V^{l+1} . This step is implicit in direction *x*. In a second step, defined by Equation (24), we use $V^{l+1/2}$ to calculate V^l . This step is implicit in the direction of *y*. We need boundary conditions to apply the algorithm, which we consider as follows:

- if x = 0 then the payoff function is 0 and so the option price is 0.
- if y = 0 then $S_2 = 0$, and so is just the price of the option on one risky asset.

Note that due to the use of centred approximations of the derivatives, at $x_0 = y_0 = 0$, $x_M = x_{max}$ and $y_N = y_{max}$, there appear external fictitious nodes $x_{-1} = -\Delta x$, $y_{-1} = -\Delta y$, $x_{M+1} = (M + 1)\Delta x$ and $y_{N+1} = (N + 1)\Delta y$. The approximations in these nodes are obtained by using linear interpolation throughout the approximations obtained in the closest interior nodes of the numerical domain. Thus we have the following relations:

$$V_{-1,n}^{l} = 2V_{0,n}^{l} - V_{1,n}^{l}, \quad V_{M+1,n}^{l} = 2V_{M,n}^{l} - V_{M-1,n}^{l}, \quad n = 0(1)N,$$

$$V_{m,-1}^{l} = 2V_{m,0}^{l} - V_{m,1}^{l}, \quad V_{m,N+1}^{l} = 2V_{m,N}^{l} - V_{m,N-1}^{l}, \quad m = 0(1)M,$$
(25)

and also

$$V_{-1,N+1}^{l} = 4V_{0,N}^{l} - 2(V_{1,N}^{l} + V_{0,N-1}^{l}) + V_{1,N-1}^{l},$$

$$V_{M+1,-1}^{l} = 4V_{M,0}^{l} - 2(V_{M,1}^{l} + V_{N-1,0}^{l}) + V_{N-1,1}^{l},$$

$$V_{-1,-1}^{l} = 4V_{0,0}^{l} - 2(V_{0,1}^{l} + V_{1,0}^{l}) + V_{1,1}^{l},$$

$$V_{M+1,N+1}^{l} = 4V_{M,N}^{l} - 2(V_{M,N-1}^{l} + V_{M-1,N}^{l}) + V_{M-1,N-1}^{l}.$$
(26)

Now all values $V_{m,n}^l$ are available. By repeating this procedure for l = L - 1, L - 2, ..., 0, we obtain $V_{m,n}$ at all time points and can approximate the price of a Spread option at time t = 0.

3.2 Stability and convergence of the numerical solution

In this section, we analyse stability of the Peaceman-Rachford method. In this case, we can use the Von Neumann analysis to establish the conditions of stability. This approach was described in Chapter 2.2 of [38]. The Von Neumann analysis is based on calculating the amplification factor of a scheme, g, and deriving conditions under which $|g| \le 1$. For finding the amplification factor, a simpler and equivalent procedure is to replace V_{mn}^l in the scheme by $g^{-l} e^{im\theta} e^{in\phi}$ for each value of *l*, *n* and *m*. The resulting equation can then be solved for the amplification factor. Replacing $V_{mn}^{l+1/2}$ and V_{mn}^{l} by $\hat{g}g^{-l}e^{im\theta}e^{in\phi}$ and $g^{-l}e^{im\theta}e^{in\phi}$, respectively, we have

$$\frac{\Delta t}{2} A_{dx} V_{m,n}^{l+1/2} = \hat{g} g^{-l} e^{im\theta} e^{in\phi} \left(-a_1 \sin^2 \frac{1}{2} \theta + b_1 i \sin \theta \right),$$

$$\frac{\Delta t}{2} A_{dy} V_{m,n}^l = g^{-l} e^{im\theta} e^{in\phi} \left(-a_2 \sin^2 \frac{1}{2} \phi + b_2 i \sin \phi - c_1 \right),$$

$$\frac{\Delta t}{2} A_{dx \, dy} V_{m,n}^{0,l+1/2} = -\hat{g} g^{-l} e^{im\theta} e^{in\phi} c_2^{l+1/2} \sin \theta \sin \phi,$$

$$\frac{\Delta t}{2} A_{dx \, dy} V_{m,n}^{0,l} = -g^{-l} e^{im\theta} e^{in\phi} c_2^l \sin \theta \sin \phi,$$

(27)

where

$$a_{1} \coloneqq a_{1}(x_{m}, y_{n}, t_{l+1/2}) = \frac{\Delta t(\sigma_{1}^{2}x_{m}^{2} + \lambda^{2}\sigma_{2}^{2}y_{n}^{2}(V_{xy}^{BS}(x_{m}, y_{n}, t_{l+1/2}))^{2} + 2\rho\sigma_{1}\sigma_{2}x_{m}y_{n}\lambda V_{xy}^{BS}(x_{m}, y_{n}, t_{l+1/2}))}{\Delta x^{2}(1 - \lambda V_{xx}^{BS}(x_{m}, y_{n}, t_{l+1/2}))^{2}},$$

$$b_{1} \coloneqq b_{1}(x_{m}) = \frac{\Delta trx_{m}}{2\Delta x},$$

$$b_{2} \coloneqq b_{2}(y_{n}) = \frac{\Delta try_{n}}{2\Delta y},$$

$$a_{2} \coloneqq a_{2}(y_{n}) = \frac{\Delta t\sigma_{2}^{2}y_{n}^{2}}{\Delta y^{2}},$$

$$c_{1} = \frac{r\Delta t}{2},$$
(28)

and

$$c_{2}^{l} =: c_{2}(x_{m}, y_{n}, t_{l}), \quad c_{2}^{l+1/2} =: c_{2}(x_{m}, y_{n}, t_{l+1/2}), \quad c_{2}^{l+1} =: c_{2}(x_{m}, y_{n}, t_{l+1}),$$

$$c_{2}(x_{m}, y_{n}, t_{l}) = \frac{\Delta t}{2\Delta x \Delta y (1 - \lambda(x_{m}, t_{l}) V_{xx}^{BS}(x_{m}, y_{n}, t_{l}))} (\sigma_{1} \sigma_{2} \rho x_{m} y_{n} + \lambda(x_{m}, t_{l}) \sigma_{2}^{2} y_{n}^{2} V_{xy}^{BS}(x_{m}, y_{n}, t_{l})).$$
(29)

We obtain the amplification factor as

$$g = \frac{1 - a_2 \sin^2 \frac{1}{2}\phi + b_2 i \sin \phi - c_1 - c_2^{l+1} \sin \theta \sin \phi}{(1 + a_1 \sin^2 \frac{1}{2}\theta - b_1 i \sin \theta)\hat{g}},$$
(30)

where

$$\hat{g} = \frac{1 + a_2 \sin^2 \frac{1}{2}\phi - b_2 i \sin \phi + c_1}{1 - a_1 \sin^2 \frac{1}{2}\theta + b_1 i \sin \theta - c_2^{l+1/2} \sin \theta \sin \phi},$$
(31)

by arranging, we have

$$g = \frac{[1 - a_1 \sin^2 \frac{1}{2}\theta - c_2^{l+1/2} \sin \theta \sin \phi + (b_1 \sin \theta)i]}{[1 - a_2 \sin^2 \frac{1}{2}\phi - c_1 - c_2^{l+1} \sin \theta \sin \phi + (b_2 \sin \phi)i]}{[1 + a_1 \sin^2 \frac{1}{2}\theta - (b_1 \sin \theta)i][1 + a_2 \sin^2 \frac{1}{2}\phi + c_1 - (b_2 \sin \phi)i]},$$
(32)

and thus

$$|g(\theta,\phi)|^{2} = \frac{[(1-a_{1}\sin^{2}\frac{1}{2}\theta - c_{2}^{l+1/2}\sin\theta\sin\phi)^{2} + b_{1}^{2}\sin^{2}\theta]}{[(1-a_{2}\sin^{2}\frac{1}{2}\phi - c_{1} - c_{2}^{l+1}\sin\theta\sin\phi)^{2} + b_{2}^{2}\sin^{2}\phi]}$$
(33)

Since $\lambda(t, S_1(t)) = \varepsilon \hat{\lambda}(t, S_1(t))$ the coefficients $a_1, c_2', c_2^{l+1/2}$ and c_2^{l+1} are continuous with respect to ε . Therefore the amplification factor g is continuous with respect to ε as well. Thus if |g| < 1 for $\varepsilon_0 = 0$ then there is a neighbourhood B_{ε_0} of ε_0 such that $|g| \le 1$ for all $\varepsilon \in B_{\varepsilon_0}$. Moreover $a_1 > 0$ for $\varepsilon_0 = 0$ so $a_1 \ge 0$ for all $\varepsilon \in B_{\varepsilon_0}$.

For $\varepsilon = 0$, according to definitions (28) and (29), $a_2 = Ca_1, c_2^l = c_2^{l+1/2} = c_2^{l+1} = \hat{C}a_1$, where $C =: C(m, n) = (\sigma_2 n / \sigma_1 m)^2, \hat{C} =: \hat{C}(m, n) = \rho \sigma_2 n / 2 \sigma_1 m$. Since $|\rho| \le 1$ it follows that

$$C \ge 4\hat{C}^2. \tag{34}$$

Moreover $b_1 = \xi a_1, b_2 = (\xi n/m)a_1, c_1 = (\xi/m)a_1$, where $\xi =: r/2\sigma_1^2 m$. By replacing the above relations in Equation (33), we find out that

$$\lim_{\xi \to 0} g(\theta, \phi)^2 = \frac{(1 - a_1 \sin^2 \frac{1}{2}\theta - \hat{C}a_1 \sin \theta \sin \phi)^2 (1 - Ca_1 \sin^2 \frac{1}{2}\phi - \hat{C}a_1 \sin \theta \sin \phi)^2}{(1 + a_1 \sin^2 \frac{1}{2}\theta)^2 (1 + Ca_1 \sin^2 \frac{1}{2}\phi)^2}.$$
 (35)

Hence it is enough to find the conditions for which

$$\frac{(1-a_1\sin^2\frac{1}{2}\theta - \hat{C}a_1\sin\theta\sin\phi)^2(1-Ca_1\sin^2\frac{1}{2}\phi - \hat{C}a_1\sin\theta\sin\phi)^2}{(1+a_1\sin^2\frac{1}{2}\theta)^2(1+Ca_1\sin^2\frac{1}{2}\phi)^2} < 1.$$
 (36)

Notice that

$$a_{1} \sin^{2} \frac{1}{2}\theta + \hat{C}a_{1} \sin \theta \sin \phi \leq a_{1} |\sin^{2} \frac{1}{2}\theta| + \hat{C}a_{1} |\sin \theta \sin \phi|$$

$$\leq a_{1} |\sin \frac{1}{2}\theta| [|\sin \frac{1}{2}\theta| + 2\hat{C} |\cos \frac{1}{2}\theta \sin \phi|] \qquad (37)$$

$$\leq a_{1} [1 + 2\hat{C}].$$

Thus $1 - a_1 \sin^2 \frac{1}{2}\theta - \hat{C}a_1 \sin \theta \sin \phi \ge 0$, provided that $a_1[1 + 2\hat{C}] \le 1$, and

$$Ca_{1} \sin^{2} \frac{1}{2}\phi + \hat{C}a_{1} \sin \theta \sin \phi \leq Ca_{1} |\sin^{2} \frac{1}{2}\phi| + \hat{C}a_{1} |\sin \theta \sin \phi|$$

$$\leq a_{1} |\sin \frac{1}{2}\phi| [C \sin \frac{1}{2}\phi + 2\hat{C} |\cos \frac{1}{2}\phi \sin \theta|] \qquad (38)$$

$$\leq a_{1} [C + 2\hat{C}].$$

Thus $1 - Ca_1 \sin^2 \frac{1}{2}\phi - \hat{C}a_1 \sin \theta \sin \phi \ge 0$, provided that $a_1[C + 2\hat{C}] \le 1$. Now we should find the conditions under which

$$\frac{(1-a_1\sin^2\frac{1}{2}\theta - \hat{C}a_1\sin\theta\sin\phi)(1 - Ca_1\sin^2\frac{1}{2}\phi - \hat{C}a_1\sin\theta\sin\phi)}{(1+a_1\sin^2\frac{1}{2}\theta)(1 + Ca_1\sin^2\frac{1}{2}\phi)} < 1,$$
 (39)

or equivalently

$$a_1(\sin^2\frac{1}{2}\theta + \hat{C}\sin\theta\sin\phi + C\sin^2\frac{1}{2}\phi)(-2 + a_1\hat{C}\sin\theta\sin\phi) < 0.$$

$$\tag{40}$$

If $|y| \le 1$, then for any $x \in R$, $xy \ge -|x|$, and by Equation (34)

$$\sin^{2} \frac{1}{2}\theta + \hat{C}\sin\theta\sin\phi + C\sin^{2} \frac{1}{2}\phi \ge |\sin \frac{1}{2}\theta|^{2} - 4\hat{C}|\sin \frac{1}{2}\theta\sin\frac{1}{2}\phi| + 4\hat{C}^{2}|\sin\frac{1}{2}\phi|^{2}$$

$$= (|\sin \frac{1}{2}\theta| - 2\hat{C}|\sin\frac{1}{2}\phi|)^{2} \ge 0.$$
(41)

Hence Equation (40) is satisfied if $a_1 < 2/\hat{C}$. Consequently a sufficient condition for the amplification factor to be bounded by 1, i.e. $|g(\theta, \phi)| \le 1$, is

$$a_1 < A = \min\left\{\frac{2}{\hat{C}}, \frac{1}{1+2\hat{C}}, \frac{1}{4\hat{C}^2+2\hat{C}}\right\} \quad \text{or} \quad \frac{\Delta t}{\Delta x^2} \le \frac{A}{\sigma_1^2 \cdot x_{\max}^2}, \quad \frac{\Delta t}{\Delta y^2} \le \frac{A}{\sigma_2^2 \cdot y_{\max}^2}.$$
 (42)

Although a_1 involves partial derivatives of V^{BS} the first condition can be met for $\varepsilon \in B_{\varepsilon_0}$. By assuming $\Delta x = \Delta y$ and $x_{\max} = y_{\max}$, a sufficient condition for the stability of the scheme is

$$\frac{\Delta t}{\Delta x^2} \le \frac{A}{\max\{\sigma_1^2, \sigma_2^2\} x_{\max}^2}.$$
(43)

Thus, the Peaceman–Rachford scheme is stable if the number of steps in the time interval, L, and in the spatial domain, M = N, satisfy inequality (43). This condition is a consequence of the cross-derivative term in the formula for the amplification factor. In the absence of this term, the scheme would be unconditionally stable.

The remaining issue we need to address is the convergence of the numerical method. According to [38] the scheme is consistent and hence the scheme is convergent. Numerical results of this convergence are investigated in the next section. Notice that according to [38] the scheme has first-order accuracy in time and second order in space. The Peaceman-Rachford scheme in the absence of the cross-derivative term defines an unconditionally stable scheme with a higher-order of accuracy $[O(\Delta t^2) + O(\Delta x^2)]$ (dependent on Θ of Equation (13)). However, in the presence of the mixed derivatives, the accuracy remains $[O(\Delta t) + O(\Delta x^2)]$ independent of Θ . Although the higher-order accuracy leads to a more efficient method, the numerical results in next section show the efficiency of the scheme. Modified schemes which overcome this restriction attain a higher-order of accuracy (at least $O(\Delta t^2)$). Craig and Sneyd [7] developed a ADI scheme the so-called CS scheme for parabolic equation with mixed derivatives to attain a stable second-order ADI scheme; Walfert [40] modified the CS scheme and introduced modified Craig–Snevd (MCS) to obtain the unconditional stability of second-order ADI schemes in the numerical solution of finite difference discretization of multi-dimensional diffusion problems containing mixed spatialderivative terms; Hundsdorfer [18] and Hundsdorfer and Verwer [19] presented the HV scheme for numerical solution of time-dependent advection-diffusion-reaction equations.

4. Numerical results

In this section, we provide numerical results of the partial liquidity effect in the underlying asset market. We fix the values of the parameters of the marginal dynamical equations according to Table 1.

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	$S(t_0)$	σ	S _{min}	S _{max}
Asset 1	100	0.15	0	200
Asset 2	100	0.10	0	200

Table 1. Model data together with r = 0.05.

Table 2. Convergence of the Peaceman–Rachford method introduced in Section 3 for a call exchange option in standard Black–Scholes model, based on different correlation and expiration date.

	т	l	T = 0.1	T = 0.3	T = 0.5	T = 0.7	T = 1
$\rho = 0.1$	50	100	2.0118	3.6578	4.7508	5.6265	6.7160
	100	100	2.1298	3.7181	4.7970	5.6655	6.7487
	200	200	2.1555	3.7327	4.8083	5.6750	6.7566
Margrabe			2.1665	3.7516	4.8422	5.7279	6.8436
$\rho = 0.5$	50	100	1.5860	2.8475	3.6841	4.3553	5.1912
	100	100	1.6544	2.8723	3.7012	4.3692	5.2029
	200	200	1.6641	2.8773	3.7053	4.3730	5.2065
Margrabe			1.6686	2.8900	3.7304	4.4132	5.2737
$\rho = 0.7$	50	100	1.3525	2.3746	3.0464	3.5849	4.2560
,	100	210	1.3733	2.3519	3.0192	3.5578	4.2308
	200	200	1.3600	2.3391	3.0087	3.5492	4.2242
Margrabe			1.3528	2.3429	3.0244	3.5782	4.2761
$\rho = 0.9$	50	100	1.1030	1.8391	2.3029	2.6697	3.1236
	100	100	1.0500	1.7175	2.1691	2.5336	2.9897
	200	200	0.9846	1.6498	2.1068	2.4766	2.9395
Margrabe	_00	200	0.9356	1.6204	2.0918	2.4750	2.9580

Note: *Margrabe* is the result of Margrabe's closed formula which appear in italic and the other numbers are the approximation solution of our method. *m* denotes the number of steps in the spatial domain, while *l* is the number of time steps. The values of the parameters used for these runs are given in Table 1.

We also assume the following price impact form

$$\lambda = \begin{cases} \varepsilon (1 - e^{-\beta (T-t)^{3/2}}), & \underline{S} \leqslant S_1 \leqslant \overline{S}, \\ 0, & \text{otherwise,} \end{cases}$$

where ε is a constant price impact coefficient, T - t is time to expiry, β is a decay coefficient, \underline{S} and \overline{S} represent, respectively, the lower and upper limit of the stock price within which there is a impact price.

We consider $\underline{S} = 60$, $\overline{S} = 140$, $\varepsilon = 0.01$ and $\beta = 100$ for the subsequent numerical analysis. Choosing a different value for β , \underline{S} and \overline{S} will change the magnitude of the subsequent results, however, the main qualitative results remain valid. At maturity T, on the line x + y = K, the BS gamma $\partial^2 V^{BS} / \partial x^2$ will blow up. However the above choice of λ guarantees that at maturity $\lambda (\partial^2 V^{BS} / \partial x^2) = 0$.

Convergence of numerical results. For the investigation of the numerical scheme, since the PDE (4) with $\lambda = 0$ is the standard Black–Scholes model, we can compare the numerical results for $\lambda = 0$ with the Margrabe's closed formula while k = 0. We fix the values of the parameters of the marginal dynamical equations according to Table 1, and vary the values of the correlation coefficient ρ . Results of this convergence study are summarized in Table 2. In comparison of the efficiency and accuracy, we can see from the table that the agreement is excellent. We plot the absolute error between our approximation and Margrabe's closed formula against the correlation in Figure 1. The numerical value of call Spread option in illiquid market is stated in Table 3. The values of the parameters used for these runs are given in Table 1, with different strike price.



Figure 1. Absolute errors between our approximation and Margrabe's closed formula. Data are in given in Table 1 with $\rho = 0.7$, T = 0.7 year, m = 50 and l = 100.

	k = -15	k = -5	k = -2	k = 0	k = 2	k = 5	k = 10	k = 20
$\rho = 0.1$	15.0923	7.1590	5.3265	4.2927	3.4018	2.3388	1.1263	0.1904
Excess price	-0.0003	-0.0004	-0.0004	-0.0003	-0.0002	-0.0001	0.0000	0.0000
$\rho = 0.5$	14.7990	6.2962	4.3635	3.3138	2.4476	1.4902	0.5431	0.0426
Excess price	-0.0001	-0.0001	-0.0001	-0.0001	0.0000	0.0000	0.0000	0.0000
$\rho = 0.7$	14.7084	5.7964	3.7719	2.7073	1.8631	0.9972	0.2589	0.0055
Excess price	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\rho = 0.9$	14.6832	5.2287	3.0505	1.9581	1.1513	0.4375	0.0085	0.0029
Excess price	0.0000	0.0001	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000

Table 3. The values of a 0.4 year European call Spread option based on different correlation, and strike price structure.

Note: Excess price shows the difference in call Spread option from Black–Scholes. The values of the parameters used for these runs are given in Table 1 with m = l = 100.

Replicating cost. Now, we are ready to investigate the effects of the partial price impact (firstorder feedback model) on the replication of Spread option. We plot the time 0 difference between the call price in the first-order feedback model and the corresponding Black–Scholes price against the stock price $S_1(0)$ and $S_2(0)$. The difference between the amount borrowed to replicate a call (in the first-order feedback model and the classical no impact model) at time 0 with expiration date T = 0.1, 0.4 and 1 year, are shown, respectively, in Figures 2–4. The figures indicate that, the Spread option price in the first-order feedback model is less than the classical Spread option price. In other words, the trader can borrow less (for a call) or lend less (for a put) to replicate a call or put Spread option.

Excess cost. Figure 5 shows the numerical results from the excess replicating costs above the corresponding Black–Scholes price (obtained using the Peacman–Rachford scheme with m = l = 100) for a call as a function of the strike price (with $S_1(t_0) = 100$, $S_2(t_0) = 110$, $\sigma_1 = 0.15$, $\sigma_2 = 0.10$, r = 0.05, $\rho = 0.7$, T = 0.4 year). As the option becomes more and more in the money



Figure 2. The call price difference (first-order feedback model and classical model) as a function of stock price at time 0 against S_1 and S_2 . K = 5, $\sigma_1 = 0.3$, $\sigma_2 = 0.2$, r = 0.05, $\rho = 0.7$, T = 0.1 and m = l = 100.



Figure 3. The call price difference (first-order feedback model and classical model) as a function of stock price at time 0 against S_1 and S_2 . K = 5, $\sigma_1 = 0.3$, $\sigma_2 = 0.2$, r = 0.05, $\rho = 0.7$, T = 0.4 and m = l = 100.

and out of the money, the excess cost decreases and converges monotonically to zero. However, as the option gets more and more out of the money, the trader needs to buy less stock and eventually, when the option is far in the money and out of the money, the investor does not need to buy any share.



Figure 4. The call price difference (first-order feedback model and classical model) as a function of stock price at time 0 against S_1 and S_2 . K = 5, $\sigma_1 = 0.3$, $\sigma_2 = 0.2$, r = 0.05, $\rho = 0.7$, T = 1 and m = l = 100.



Figure 5. The replicating cost difference (first-order feedback model and classical model) against the strike price *K*. $S_1(t_0) = 100$, $S_2(t_0) = 110$, $\sigma_1 = 0.15$, $\sigma_2 = 0.10$, r = 0.05, $\rho = 0.7$, T = 0.4 year and m = l = 100.

5. Conclusion

In this work, we have investigated a model which incorporates illiquidity of the underlying asset into the classical multi-asset Black–Scholes–Merton framework. We considered the first-order feedback model in which only a large trader affect the underlying price and the trading strategies of other traders do not influence the price. Since there is no analytical formula for the price of a option within this model, we proposed the partial differential equation approach to price options. We applied a standard ADI method (Peaceman–Rachford scheme) to solve the partial differential equation numerically. We also discussed the stability and the convergence of the numerical scheme. By numerical experiment, we investigated the effects of liquidity on the Spread option pricing in the first-order feedback model. As future research we plan to investigate other schemes (including CS, MCS and HV) and their stability.

Finally, we found out that the Spread option price in the market with finite liquidity (firstorder feedback model) is less than the Spread option price in the classical Black–Scholes–Merton framework. Consequently one needs to borrow less (for a call) or lends less (for a put) to replicate a call or put in a first-order feedback model.

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References

- J. Back and M. Prokopczuk, *Commodity price dynamics and derivatives valuation: A Review*, Working paper, Zeppelin University, 2012. Available at SSRN 2133158.
- [2] F. Black and M. Scholes, The pricing of options and corporate liabilities, J. Political Econ. 81(3) (1973), pp. 637–654.
- [3] M.J. Brennan and E.S. Schwartz, Evaluating natural resource investments, J. Bus. 58 (1985), pp. 135–157.
- [4] G. Cortazar and E.S. Schwartz, A compound option model of production and intermediate inventories, J. Bus. 66(4) (1993), pp. 517–540.
- [5] G. Cortazar and E.S. Schwartz, The valuation of commodity contingent claims, J. Derivatives 1(4) (1994), pp. 27–39.
- [6] J.C. Cox and E.S. Schwartz, The pricing of commodity linked bonds, J. Financ. 37(2) (1982), pp. 525–539.
- [7] I.J.D. Craig and A.D. Sneyd, An alternating-direction implicit scheme for parabolic equations with mixed derivatives, Comput. Math. Appl. 16(4) (1988), pp. 341–350.
- [8] D.M. Dang, C. Christara, K. Jackson, and A. Lakhany, An efficient numerical PDE approach for pricing foreign exchange interest rate hybrid derivatives, preprint (2012). Available at SSRN 2028519.
- [9] D. Duffie, Dynamic Asset Pricing Theory, Princeton University Press, Princeton, 2010.
- [10] D.J. Duffy, Finite Difference Methods in Financial Engineering: A Partial Differential Equation Approach, John Wiley & Sons, Chichester, 2006.
- [11] G. Fichera, Sulle equazioni differenziali lineari ellittico-paraboliche del secondo ordine, Atti della Accademia Nazionale dei Lincei. Memorie. Classe di Scienze Fisiche, Matematiche e Naturali, Serie VIII (in Italian) 5 (1) (1956), pp. 1–30.
- [12] R. Frey, Perfect option hedging for a large trader, Financ. Stoch. 2(2) (1998), pp. 115–141.
- [13] R. Gibson and E.S. Schwartz, Stochastic convenience yield and the pricing of oil contingent claims, J. Financ. 45(3) (1990), pp. 959–976.
- [14] P.B. Girma and A.S. Paulson, Seasonality in petroleum futures spreads, J. Futures Markets 18(5) (1998), pp. 581–598.
- [15] P.B. Girma and A.S. Paulson, *Risk arbitrage opportunities in petroleum futures spreads*, J. Futures Markets 19(8) (1999), pp. 931–955.
- [16] K.J. Glover, P.W. Duck, and D.P. Newton, On nonlinear models of markets with finite liquidity: Some cautionary notes, SIAM J. Appl. Math. 70(8) (2010), pp. 3252–3271.
- [17] T. Haentjens, ADI finite difference schemes for the Heston-Hull-White PDE. No. 1111.4087, preprint (2011). Available at arXiv.org.
- [18] W. Hundsdorfer, Accuracy and stability of splitting with stabilizing corrections, Appl. Numer. Math. 42(1) (2002), pp. 213–233.
- [19] W. Hundsdorfer and J.G. Verwer, Numerical Solution of Time-dependent Advection–Diffusion-Reaction Equations, Vol. 33, Springer, Berlin, 2003.
- [20] K.J. In't Hout and S. Foulon, ADI finite difference schemes for option pricing in the Heston model with correlation, Int. J. Numer. Anal. Model. 7(2) (2010), pp. 303–320.

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- [21] L. Jódar and J.-R. Pintos, A consistent stable numerical scheme for a nonlinear option pricing model in illiquid markets, Math. Comput. Simul. 82(10) (2012), pp. 1972–1985.
- [22] R.L. Johnson, C.R. Zulauf, S.H. Irwin, and M.E. Gerlow, *The soybean complex spread: An examination of market efficiency from the viewpoint of a production process*, J. Futures Markets 11(1) (1991), pp. 25–37.
- [23] E. Kirk, Correlation in the energy markets, in Managing Energy Price Risk, Risk Publications and Enron, London, 1995, pp. 71–78.
- [24] A. Lipton, Mathematical Methods for Foreign Exchange: A Financial Engineer's Approach, World Scientific Publishing Company Incorporated, Singapore, 2001.
- [25] H. Liu and J. Yong, Option pricing with an illiquid underlying asset market, J. Econ. Dyn. Control 29(12) (2005), pp. 2125–2156.
- [26] W. Margrabe, The value of an option to exchange one asset for another, J. Financ. 33(1) (1978), pp. 177-186.
- [27] S. McKee and A.R. Mitchell, Alternating direction methods for parabolic equations in two space dimensions with a mixed derivative, Comput. J. 13(1) (1970), pp. 81–86.
- [28] S. McKee, D.P. Wall, and S.K. Wilson, An alternating direction implicit scheme for parabolic equations with mixed derivative and convective terms, J. Comput. Phys. 126(1) (1996), pp. 64–76.
- [29] New York Mercantile Exchange, Crack Spread Handbook, New York Mercantile Exchange, 2011. Available at partners.futuresource.com/marketcenter/pdfs/crack.pdf.
- [30] J.L. Paddock, D.R. Siegel, and J.L. Smith, Option valuation of claims on real assets: The case of offshore petroleum leases, Quart. J. Econ. 103(3) (1988), pp. 479–508.
- [31] R. Paschke and M. Prokopczuk, Commodity derivatives valuation with autoregressive and moving average components in the price dynamics, J. Bank. Financ. 34(11) (2010), pp. 2742–2752.
- [32] D.W. Peaceman and H.H. Rachford Jr, *The numerical solution of parabolic and elliptic differential equations*, J. Soc. Ind. Appl. Math. 3(1) (1955), pp. 28–41.
- [33] L. Pospisil and J. Vecer, PDE methods for the maximum drawdown, J. Comput. Financ. 12(2) (2008), pp. 59-76.
- [34] C. Randall, PDE Techniques for Pricing Derivatives with Exotic Path Dependencies or Exotic Processes, Lecture Notes, Workshop CANdiensten, Amsterdam, 2002.
- [35] E.S. Schwartz, The stochastic behavior of commodity prices: Implications for valuation and hedging, J. Financ. 52(3) (1997), pp. 923–973.
- [36] E. Schwartz and J.E. Smith, Short-term variations and long-term dynamics in commodity prices, Manage. Sci. 46(7) (2000), pp. 893–911.
- [37] A. Shidfar, Kh. Paryab, A.R. Yazdanian, and T.A. Pirvu, Numerical analysis for Spread option pricing model in illiquid underlying asset market: full feedback model (under review).
- [38] J. Strikwerda, Finite Difference Schemes and Partial Differential Equations, SIAM, Philadelphia, PA, 2007.
- [39] B.D. Welfert, Stability of ADI schemes applied to convection-diffusion equations with mixed derivative terms, Appl. Numer. Math. 57(1) (2007), pp. 19–35.
- [40] B.D. Welfert, Unconditional stability of second-order ADI schemes applied to multi-dimensional diffusion equations with mixed derivative terms, Appl. Numer. Math. 59(3) (2009), pp. 677–692.
- [41] P. Wilmott and P.J. Schönbucher, The feedback effect of hedging in illiquid markets, SIAM J. Appl. Math. 61(1) (2000), pp. 232–272.
- [42] T.P. Witelski and M. Bowen, ADI schemes for higher-order nonlinear diffusion equations, Appl. Numer. Math. 45(2) (2003), pp. 331–351.