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Numerical Methods for Finance Coursework

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1 Introduction

We have seen that in general there is no arbitrage if and only if there exists an equivalent local martingale measure. In this case, the price of an asset is the discounted price of the payoff under this new probability.

In most of the models that we have seen, such as Black Scholes, the discounted underlying asset has always been a martingale under the pricing probability.
However, what does happen if this asset was instead a strict local martingale ?
It has been shown that in these cases, the uniqueness of the solution of the Black-Scholes equation is no longer verified. The difference of the multiple solutions is termed as financial bubble and the option price that allows no arbitrage may be one of these solutions. Numerically, our problem is to find a numerical solution that converges to the option price under the lack of uniqueness of Black Scholes PDE.

2 Problem Formulation

Consider that the model of the stock price X under the pricing measure is as follow (assuming interest rate is equal to 0) :

$$dX_t = \alpha(X_t, t)dW$$

with α locally Hölder continuous with exponent 1/2 satisfying $\alpha(x) > 0$ for all x $\in \mathbb{R}^+$ and

$$\alpha(0) = 0$$

Denoting g a non-negative payoff function (we consider pay-offs of at most linear growth only) , the price of the option is given by $u(X_t, t)$ where

$$u(x,t) = E_{xt}g(X_T) \tag{1}$$

It is a solution to the Black-Scholes equation given by :

$$\begin{cases} u_t(x,t) + \frac{1}{2}\alpha^2(x)u_{xx}(x,t) = 0 \text{ if (x,t)} \in (0,\infty) * [0,T) \\ u(x,T) = g(x) \\ u(0,t) = g(0) \end{cases} \tag{2}$$

And it is the unique one when $\alpha(X_t, t)$ is of at most linear growth.
However, that doesn't hold anymore as soon as the linear bound of $\alpha(X_t, t)$ is violated. Indeed, in this case, the stock price is a strict local martingale and therefore there are multiple solutions to the Black Scholes equation of at most linear growth.

Example :

Over all this coursework, we will use the Constant Elasticity of Variance (CEV) model, which gives the stock price as :

$$dX_t = \sigma X_t^{\gamma} dW$$

Where $\gamma \geq 0$ and $\sigma > 0$

If $\gamma > 1$ then X is a strict local martingale.

In this case and when g≡id the price of the option is always given by

$$u(x,t) = E_{xt} X_T$$

which is solution to (1) and $u(x,t) < x$ because X is a strict positive local martingale and hence a supermartingale. On the other hand, $u = x$ is clearly also a solution to (2). Hence uniqueness doesn't hold anymore. This underlines our main problem and the paper [1] and [2] suggest methods to find the no arbitrage price of the asset numerically despite the lack of uniqueness.

Most numerical methods are set up on a finite grid and determines a unique solution to Black-Scholes PDE thanks to the boundary condition. Then, we have to find boundary conditions that give a solution that converge to the no arbitrage price and with a good rate of convergence.

The two articles give different ways to tackle this issue :

- Add a Neumann condition at the spatial boundary
- Consider a class of rebate barrier option to approximate the option price

3 Numerical Methods

In both of these methods, several results have been used to prove that the numerical solution converges to the no arbitrage price solution.

Assumptions

(A-1): The payoff is non negative

(A-2): The payoff is of at most linear growth i.e it is a measurable function that verify $\varphi(x,t) \leq K(1 + |x|^{\epsilon})$, $\forall (x,t) \in Q := \mathbb{R}^{+} * (0,T)$ with $\epsilon \leq 1$

(A-3): The payoff is non decreasing

3.1 Neumann Boundary Conditions

Proposition 3.1.1 Assume (A-1) and (A-2). The option price u given by (1) is the smallest non-negative classical solution of (2)

To take this proposition into account, we will take another characterization of u by considering this new partial differential equation (u^M denotes its unique solution). What changes is that we add a Neumann boundary space condition where x=M.

$$\begin{cases} u_t^M(x,t) + \frac{1}{2}\alpha^2(x)u_{xx}^M(x,t) = 0 \text{ if (x,t)} \in (0,M) * [0,T) \\ u^M(x,T) = g(x) \text{ if x} \in [0,M] \\ u^M(0,t) = g(0) \text{ if t} \in [0,T) \\ u_x^M(M,t) = 0 \text{ if t} \in [0,T) \end{cases} \tag{3}$$

We extend the domain of definition by setting $u^M(x,t) = u^M(M,t)$ when $x \geq M$.

Theorem 3.1.2 Assume (A-1), (A-2) and (A-3). Then u^M is increasing in M, and:

$$u(x,t) = \lim_{M \to +\infty} u^M(x,t)$$

Sketch of the proof

- Show that u^M is increasing : The monotonicity of g implies that $u_t^M(x)$ is non-decreasing. Then use the Maximum principle.
- Define

$$\bar{u}(x,t) := \lim_{M \to +\infty} u^M(x,t)$$

We know that $u^M \leq u$ for all M (Maximun principle) so $\bar{u} \leq u$. Now we know that $\bar{u}$ (interior Schauder estimates) solves (3) and that it is non-negative and smaller than u. Apply proposition 3.1.1.

In this way, thanks to this new boundary condition, we got a solution that is convergent to the no arbitrage price of the option given in (1).

3.2 Considering rebate options

Another way to solve the problem is to consider up-rebate options of barrier M with :
- a rebate payoff $g(M)$
- a terminal payoff $f(X(T))$

That gives us the rebate option price :

$$u^M(x,t) = E_{xt}[g(M)1_{\tau^M \leqslant T} + f(X(T)1_{\tau^M = T}] \tag{4}$$

where the stopping time τ^M is the first hitting time of the stock price X to the barrier M.

We will, in fact, find the numerical price of the rebate option and show that it is convergent to the no arbitrage price (1) of the European option solving (2) when M tends to infinity when g and f are well specified.

Proposition 3.2.1

Assume (A-2) for the terminal payoff f and that the rebate payoff g is strictly sublinear. then :

$$\lim_{M \to +\infty} u^M(x,t) = u(x,t)$$

For the proof which is long, see[2]

Then the rebate option price is solution of the PDE :

$$\begin{cases} u_t(x,t) + \frac{1}{2}\alpha(x)^2 u_{xx}(x,t) = 0 \text{ on } (0,M) * [0,T) \\ u(x,T) = f(x) \text{ if x} \in [0,M] \\ u(0,t) = f(0) \text{ if t} \in (0,T) \\ u(M,t) = g(M) \text{ if t} \in (0,T) \end{cases} \tag{5}$$

The choice of $g(M) = f(M)$ may not always be possible. When we have $g(M) \neq f(M)$ and the boundary-terminal condition is discontinuous, then we can not expect the unique solution of (5) being continuous up to the boundary. Furthermore, this discontinuity and the singularity at the corner propagate the numerical errors throughout the entire domain. That is why it is crucial to consider an alternative choice for (4) by revising the terminal payoff as following to avoid this problem.

We consider:

- a zero rebate payoff $g(M) = 0$

- a revised terminal payoff defined by

$$f^M = f(x)\mathbb{1}_{x \leq M/2} + \frac{2f(x)(M-x)}{M}\mathbb{1}_{M/2 < x \leq M}$$

In this case the rebate option price becomes :

$$\tilde{u}^M(x,t) = E_{xt}[f^M(X(T))1_{\tau^M = T}] \tag{6}$$

And it is associated to PDE

$$\begin{cases} u_t(x,t) + \frac{1}{2}\alpha(x)^2 u_{xx}(x,t) = 0 \text{ on } (0,M) * [0,T) \\ u(x,T) = f^M(x) \text{ if x} \in [0,M] \\ u(0,t) = f^M(0) \text{ if t} \in (0,T) \\ u(M,t) = 0 \text{ if t} \in (0,T) \end{cases} \tag{7}$$

This revised payoff function f^M makes the terminal-boundary data continuous at the corner (M,T) and also preserves the Hölder regularity of the original payoff function f.

Although (7) is degenerate at x=0, one can show that it still has a unique solution. Moreover, this solution is given by (6) and it can be shown that this solution converges to the desired value of u. Indeed, we have the following theorem :

Theorem 3.2.2 Assume (A-1) and (A-2). Then $\tilde{u}^M$ given in (6) is the unique classical solution of PDE (7) and :

$$\lim_{M\to+\infty} \tilde{u}^M(x,t) = u(x,t)\forall(x,t) \in Q_M$$

In addition, $if\epsilon < 1$ then it can be shown that the rate of convergence is of O($M^{-1+\epsilon}$)

3.3 Implementation

3.3.1 Scheme

To implement the numerical solutions, we use the following grid:

- the space axis is divided into J intervals of length h = M / J : for j = 0,1,...,J, we define $x_j := jh$.
- the time axis is divided into N intervals of length Δt = T / N : for n = 0,1,...,T, we define $t^n := n\Delta t$.

We approximate the spatial derivative by centered second-order finite differences and we used the θ-scheme for the time-derivative. Denoting $u_j^n = u^M(x_j,t^n)$, and $u^n = (u_1^n, ...u_{J-1}^n)'$, the discretization of the PDE leads to the following equation:

$$(I_{J-1} - \theta\Delta t\frac{\alpha^2}{2h^2}A)u^n = (I_{J-1} + (1-\theta)\Delta t\frac{\alpha^2}{2h^2}A)u^{n+1}$$

where A is a tridiagonal matrix : $A = T_{J-1}(1,-2,1)$. Notice that α is a function of x, so each line j of the matrix A will be multiplied by a different coefficient α^2(j).

The vector u^n is calculated backward in time, starting from u^N (for j = 1, ..., J-1 , $u_j^N = g(x_j)$).

3.3.2 Boundary Conditions

You may have notice the absence of vector b^n as in the lecture notes in the previous scheme. Indeed to take into account the boundary conditions that we consider in this problem, it is sufficient to slightly modify the matrix A according to the case:

- the Dirichlet condition $u^M(0,t) = g(0)$ if $t \in [0,T)$ does not have any influence on the previous equation given the fact that whatever the payoff function is, it is always equal to 0 for x=0.
- the Dirichlet condition $u^M(M,t) = 0$ if $t \in (0,T)$ does not have any influence on the previous equation.
- the Neumann condition $u_x^M(M,t) = 0$ if $t \in [0,T)$ can be approximated as $u^M(M,t) = u^M(M-h,t)$, thus $u_{J-1}^n = u_J^n$ for all n. It follows that for j=J-1, and for all n, $u_{j+1}^n - 2u_j^n + u_{j-1}^n = -u_j^n + u_{j-1}^n$: in the last line of the matrix A, the "-2" becomes a "-1".

3.3.3 Choice of θ

The discretization of the PDE leads to the following conclusions:

- if $\theta \neq \frac{1}{2}$ then the accuracy in u_j^n is of O($\Delta t + h^2$)
- if $\theta = \frac{1}{2}$ (Crank-Nicolson scheme) then the accuracy in u_j^n is of O($\Delta t^2 + h^2$)

Moreover, a matrix analysis shows that the θ -scheme is unconditionally stable in the L_2-norm if $\theta \in [1/2, 1]$. Thus the choice of the Crank-Nicolson scheme could seem to be the best one. However, this stability result might not hold true in the L_∞-norm: it can be shown that if the condition $\frac{(1-\theta)\Delta t \alpha^2(x_j)}{h^2} \leq 1$ is not satisfied, the numerical solution might oscillate and even take negative values (see [5] and [6] for further discussion). An example of this instability will be shown in the results.
In order to avoid such spurious oscillations, we finally chose θ=1 (Implicit Scheme).

3.3.4 Numerical results

In this paper we will present the results obtained in Matlab for the following parameters: σ=1, γ=2 (CEV-1 model, for which a closed-form formula exists and will be taken as reference, see [4]), K=5, h=0.05, T=0.5 and Δt=0.005. h and Δt where chosen such that the error coming from the scheme was negligible in comparison to the error arising from the choice of M. For the two approaches the numerical methods has been plot for different values of M: M=$2^m 10$, m=1,2,3,4,5,6. We also computed the mean square error $\epsilon(M) := \frac{1}{J-1}\sum_{j=1}^{J-1}(u(x_j,0)-u_j^0)^2$ between the analytical and the numerical solutions.

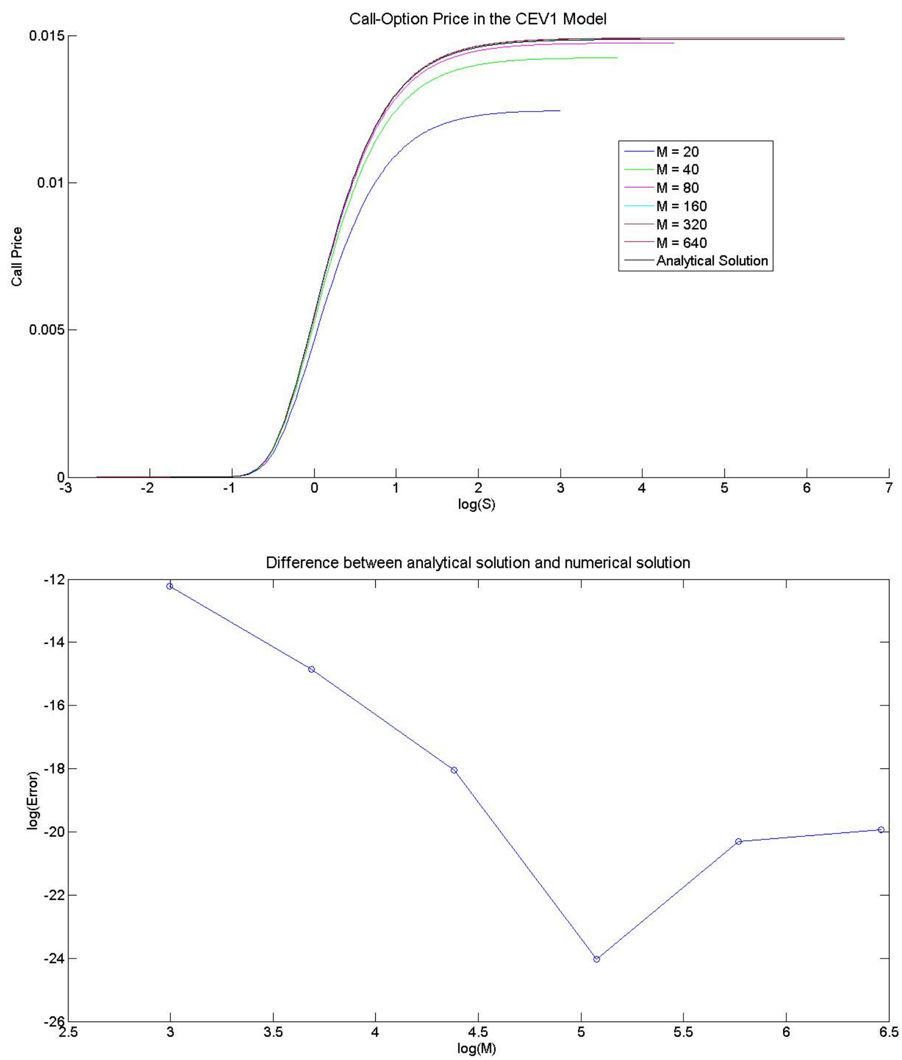

Figure 1: Solution to the Problem (3) (Neumann Boundary Condition)

As the plots suggests the numerical solution converges as M increases. However an increase in M has of course a computational cost: the computational time may become considerable. The choice of θ=1/2 led to oscillatory solutions. It is illustrated in the figures below for M=160 and two different values of σ: 0.1 and 1. As j increases the coefficient $\alpha^2(x_j)$ increases and the non-oscillatory condition is no longer satisfied. When σ is higher, the oscillations appear for smaller values of x_j. Note that to avoid this behaviour, one could use an implicit scheme for large values of x_j and the Crank-Nicolson scheme for small values in order to have a better accuracy. An other solution could be to change Δt for sensitive values of x_j.

These methods are discussed in detail in [7].

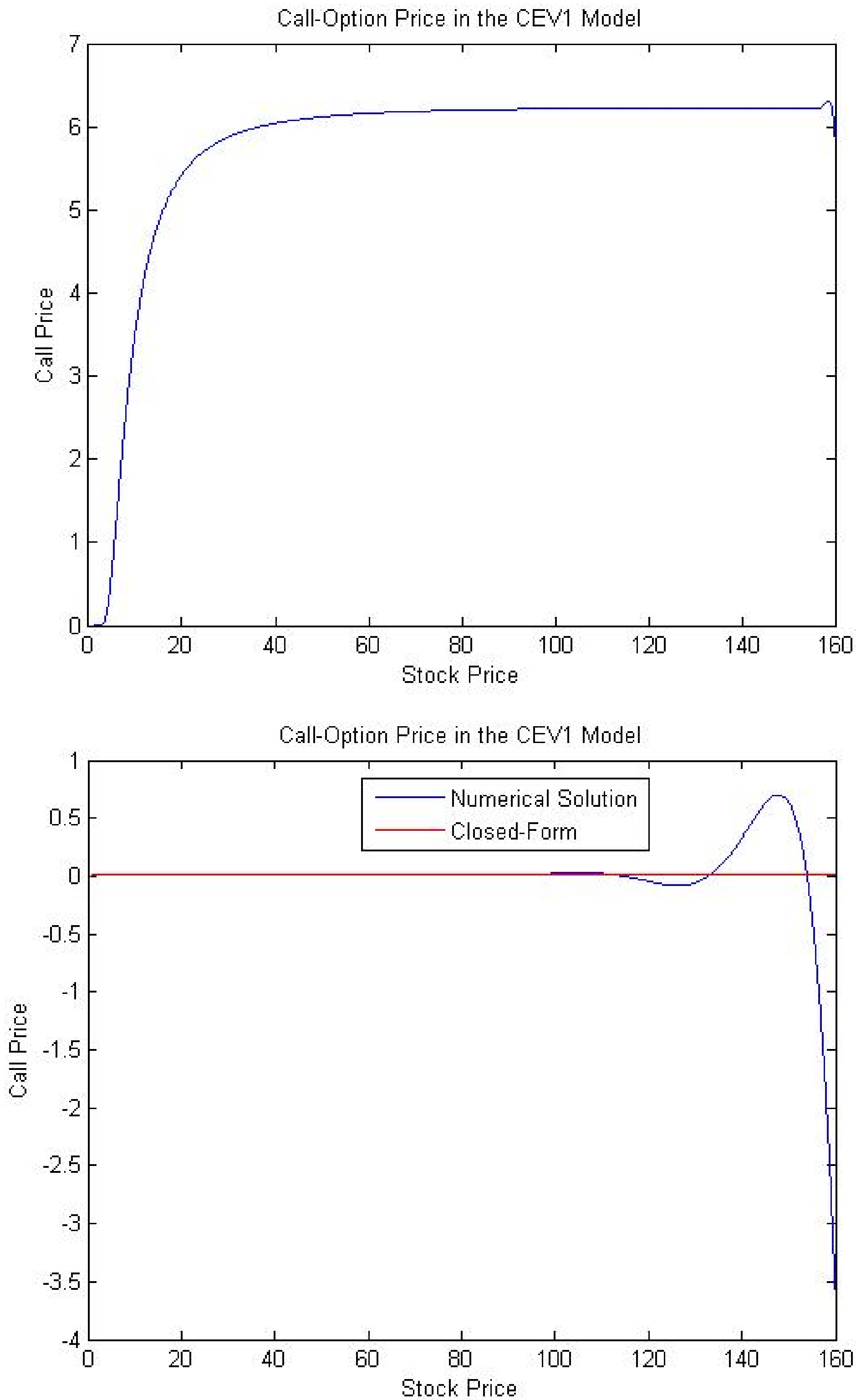

Figure 2: Illustration of the instability issue in the L_∞-norm

The second approach (Dirichlet Boundary Conditions with a rebate payoff) gave the following result:

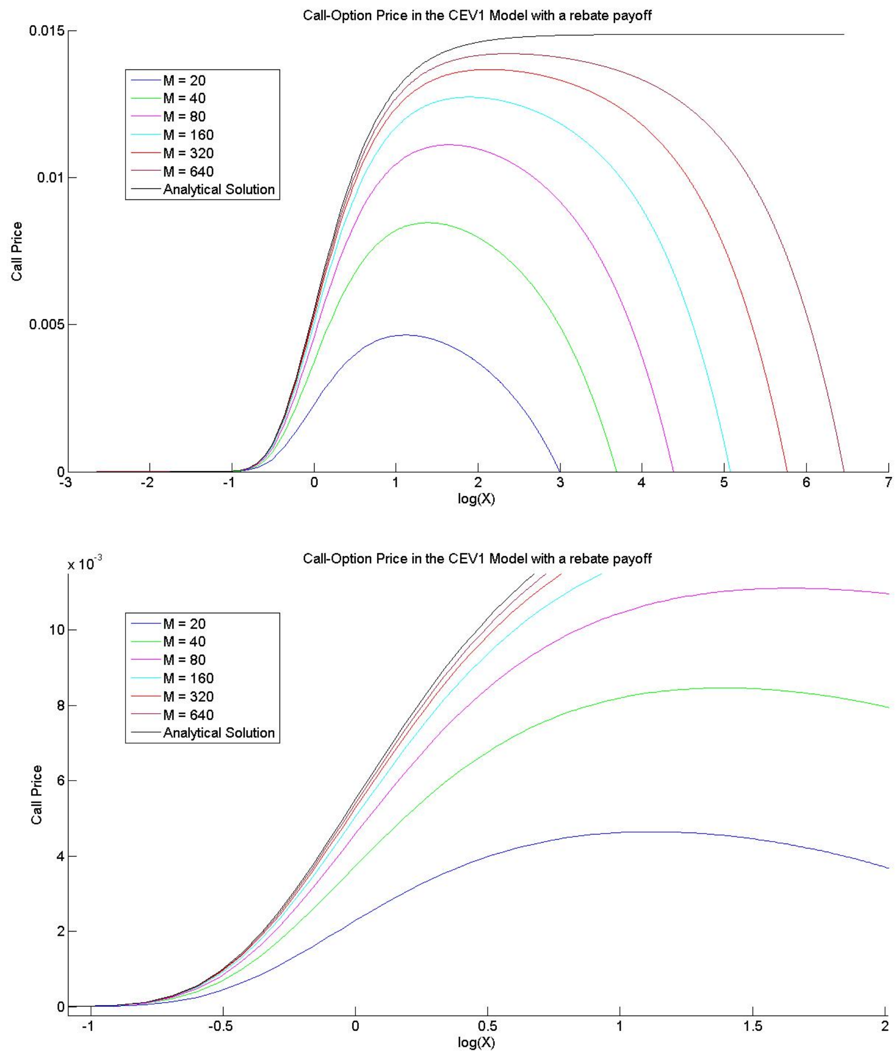

Figure 3: Solution to the Problem (4) (Dirichlet Boundary Conditions with a rebate payoff)

For small values of X the solution converges as M increases, as expected. The disadvantage of this method with regard to the first one is that the convergence as M goes to infinity is much slower. However, the specification of the boundary conditions insures that the solution is the desired one.

4 Conclusion

Through this coursework, we have seen two different methods to address the numerical vanilla option price issue in the presence of financial bubbles.These two methods give good numerical results, and could be really accurate if the time and space step are well chosen. However, as we have seen before, they both had their limits, the stability for the first one and the use of a revised terminal payoff for the second one. Moreover, these methods tackle with a different angle the issue, using different assumptions which could help to make a choice between them.

All of this shows us that only one small change in our usual assumptions could have huge effects on several models, limit that we should keep in mind when we apply them in the real world.

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