Practical Problems in the Numerical Solution of PDE's in Finance

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Abstract

In this paper we investigate the use of finite difference and finite element schemes when applied to the valuation of exotic options characterized by discontinuities in the payoff function. In particular, we will conduct a numerical analysis of several common schemes in order to give a better understanding of the numerical problems associated with the valuation of non-standard options.

KEYWORDS. Numerical Solution of PDE, Finite Difference, Finite Elements, Black-Scholes model, Discontinuous Payoff.

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1 Introduction

In this paper we investigate the application of different finite difference and finite element schemes to non-standard option pricing models. In particular we concentrate the attention on the difficulties that arise in the numerical approximation of PDE's in presence of discontinuities. Several exotic contracts fall into this situation. For example a digital option is characterized by a payoff equal to $\mathbf{1}_{(l<S<u)}$ where $\mathbf{1}_A$ stands for the indicator function

of the set A. Another example is a barrier option with a discrete monitoring clause. For example for a down-out call option, the payoff condition at maturity is continuous and equal to $(S-K)\mathbf{1}_{(K<S)}$ but the option expires worthless if before the maturity the asset price has fallen below the level l. In this case we have a discontinuity that occurs only at the monitoring dates. Moreover, also for plain vanilla options it can be shown that the sensitivities of the option price (*the greeks*) satisfy a PDE with discontinuous conditions at maturity date or even conditions involving Dirac's functions. Similar problems can occur also for more complex path-dependent derivatives securities.

In the last years, numerical techniques for solving PDE's have found a large diffusion in finance, and usually the choice goes towards a method with a high order of accuracy (e.g. Crank-Nicolson methods), and no attention is paid towards the financial provisions of the contract that can affect the reliability of the numerical solution, see for example the discussion in Zvan et al. [10]. In particular in this paper we will conduct a numerical analysis of some standard finite difference/element schemes in order to give a better understanding of the numerical problems associated with the approximation of non-standard option pricing models. A preliminary numerical analysis can give us some confidence for choosing between different methods.

We will discuss several points.

1) It is well-known that explicit schemes require prohibitively small time-step values in order to guarantee the convergence of the solution.

2) The maximum principle is the most important property for solutions to parabolic problems. For convection dominated equations, the preservation of this property in discrete analogue schemes may require a high computational cost. In fact, implicit schemes with centered differences for the first derivative term satisfy only conditionally the discrete maximum principle. Moreover discontinuous initial conditions, that introduce steep gradients in the neighboring of the barriers, may generate spurious oscillations. Unfortunately, due to the variable coefficients in option pricing models, in general the preservation of the discrete maximum principle can depend on the values of the model parameters (e.g. interest rate and volatility), independently of the discretization steps.

3) The spurious oscillations could be eliminated by appropriate modifications of centered schemes. For example the implicit upwind finite difference scheme satisfies the discrete maximum principle unconditionally. However this scheme introduces a first order numerical diffusion term that smears the numerical solution, particularly when the volatility is small or when the barriers are located in correspondence of low values of the spot price.

4) In general, Crank-Nicolson scheme is well suited for approximating parabolic pro-

blems. However, this method may suffer from spurious oscillations. For instance, in the case of barrier options, in order to achieve the desired accuracy near the barriers we should increase the spatial discretization. However this fact generates an iteration matrix characterized by negative eigenvalues with values near -1. As a consequence the numerical solution will be affected by spurious oscillations that do not decay quickly. Moreover the discontinuity will be renewed at every monitoring date. This problem could be avoided if the time-step becomes prohibitively small. We remark that the oscillations derive from an inaccurate approximation of the very sharp gradient produced by the knock-out clause, generating an error which is damped out only very slowly.

All these aspects are analysed from both a finite difference and a finite element point of view. Two different alternative methods are proposed in order to find a compromise between accuracy and computational cost of numerical schemes. The first method is a centered finite difference scheme, using a three-level time stepping which eliminates spurious oscillations by adjusting the eigenvalues of the difference matrix. The second method is a finite element approximation, which introduces a second order numerical viscosity versus the first order term of the upwind scheme, thus producing less smeared solutions.

In section 2 we present the financial problem related to the valuation of a discretely monitored barrier option. Then in section 3 we will describe various finite difference schemes and the main problems (numerical and computational) deriving from the discontinuities and from the convective-diffusive nature of the problem. Finally in section 4 we will describe the Finite Element approach and the way it can cope with discontinuities.

2 The model

We assume that the dynamics of the underlying asset is described by a standard Geometric Brownian Motion (GBM) diffusion process for the underlying:

$$dS = rSdt + \sigma SdW_t \tag{1}$$

where S is the price of the asset, r the risk-free interest rate, σ the instantaneous volatility and dW is the increment of the Brownian process.

The assumption of GBM process allows us to simplify the numerical analysis of the problem and in clarifying the problems in which we can incur. The GBM process can be relaxed for other processes such as CEV or a mean-reverting process, although greater attention will be required in choosing correctly the numerical method.

In order to make our analysis concrete we concentrate on a double barrier knock-out option, i.e. a call option that expires worthless if one of the two barriers has been hit at a monitoring date. We denote by t the time to expiry of the option and we let $0 = t_0 < t_1 < ... < t_i < ... < t_F$ be the monitoring dates and l be the lower barrier and u the upper barrier active[1] only at times t_i, $i = 1, ..., F$. Again, we are assuming constant barriers only for aim of simplicity and notational convenience. The price $V(S, t)$ of the option satisfies the Black-Scholes PDE with initial and boundary conditions:

$$-V_t + rSV_S + \frac{1}{2}\sigma^2 S^2 V_{SS} = rV \tag{2}$$

$$V(S, t_0) = (S - K)^+ \mathbf{1}_{l,u}(S) \tag{3}$$

$$V(S, t) \to 0 \text{ as } S \to 0 \text{ or as } S \to \infty \tag{4}$$

Moreover the discrete monitoring of the contract introduces an updating of the solution $V(S, t)$ at the monitoring dates $t = t_i$, $i = 1, ..., F$:

$$V(S, t_i) = V\left(S, t_i^-\right) \mathbf{1}_{l,u}(S) \tag{5}$$

where $1_{l,u}(x)$ is the indicator function:

$$\mathbf{1}_{l,u} = \begin{cases} 1 \quad if \quad l \leq x \leq u \\ \\ 0 \quad if \quad x \notin [l \;\; u] \end{cases} \tag{6}$$

Several other different contracts with discrete time monitoring are characterized by an updating condition that introduces a discontinuity, such as Parisian options and Occupation time derivatives, Fusai and Tagliani [11]. We remark that, although a sizable portion of real contracts specify fixed times for monitoring the asset, the academic research has focused mainly on continuous time monitoring models even if there can be substantial differences between discrete and continuous monitoring.

It should be noted that, away from the monitoring dates, the option price can move on the positive real axis interval $[0, +\infty)$ as illustrated in Figure 1. Then the knock-out clause at the monitoring date introduces a discontinuity at the barriers, as illustrated in Figure 2 in correspondence of different monitoring dates.

The analysis of the existence, uniqueness and regularity of the solution to (2) can be performed by different approaches. Black and Scholes [2] themselves transformed the PDE into the heat equation, whose solution is known in closed-form. Other simpler and more general transformations are possible ([4], [12]), which are capable of handling time-dependent parameters. Duffie (Appendix E) [6] solves the Black-Scholes equation (2) using

[1] Eventually, we have $l = 0$ or $u = +\infty$ in the case of single barrier options.

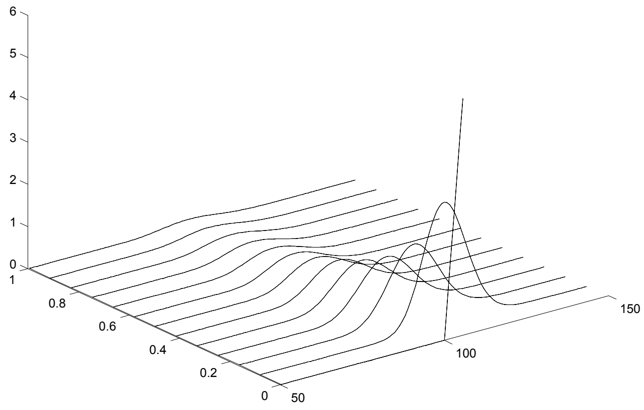

Figure 1: *Option prices just before the monitoring dates.*

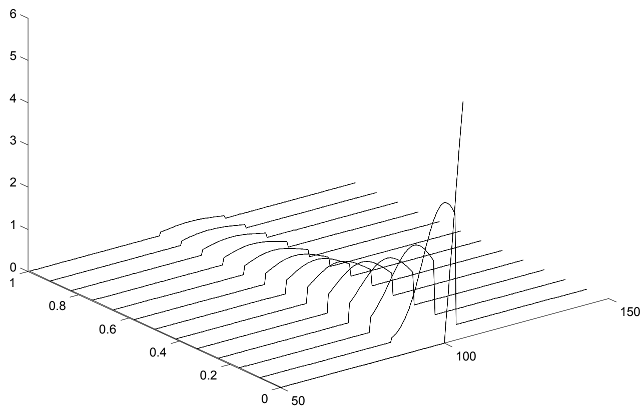

Figure 2: *Option prices just after the monitoring dates.*

Feynman-Kac theorem . Other results concerning the regularity for the solution of a class of evolution operators arising in Mathematical Finance are presented in [5], [1].

When we solve numerically the Black-Scholes equation, attention must be paid to accuracy and stability of the algorithms. Indeed, the first-order hyperbolic convective term, i.e. the term involving V_S, propagates information from the right to the left of the S axis: in financial terms it represents the increase in value of the option generated by the deterministic increase in the asset price due to the drift term. If the velocity term is large compared to the diffusive term, then the problem is said to be *convection dominated* and the PDE exhibits a hyperbolic behavior: in other words the uncertainty due to the diffusion term is negligible and we reduce to a first-order hyperbolic PDE. In this case, the numerical approximation is much harder to compute accurately, since standard numerical techniques require small discretization steps to have stability. Although this is not the case of standard European options, however, for certain path-dependent options such as Asian options, the convection dominated behavior can be extremely severe [26]. Moreover, the presence of discontinuities in the initial conditions enhances the effect of the convective term, since the solution presents steep gradients which make the approximation process more difficult.

Another source of problem in Black-Scholes equation is the presence of the awkward S and S^2 terms multiplying $\frac{\partial V}{\partial S}$ and $\frac{\partial^2 V}{\partial S^2}$ respectively. As we shall see in the following, they give rise to undesirable results in terms of the stability analysis of the different numerical schemes employed.

A standard transformation in order to eliminate both problems and reduce to a dimensionless diffusion equation ([26], page. 267) is not always convenient because a) it introduces a not equally spaced-discretization in the price space, b) the exponential functions connected with the variable transformations present very large exponents, in the presence of convection dominated models [21]. Indeed, the change of variable $S = Ke^z$ maps the S-axis $[0, +\infty)$ into the $z-$axis $(-\infty, +\infty)$. If we assume a constant step Δz in the logarithmic transformed variable and consider the corresponding distribution of nodes on the S axis, we see that they are more and more scattered as we move from 0 to infinity, since

$$S_{i+1} - S_i = K(e^{\Delta z} - 1)e^{i\Delta z} \to \infty \qquad \text{as } i \to \infty. \tag{7}$$

Many nodes are close to the origin, where the solution V is almost linear, while fewer nodes lie near the strike price K, where more accuracy is required. Finally, if we want to compute option prices at equally spaced values of S, we have to resort to *interpolation*. For instance, linear interpolation is accurate to the same order as our numerical schemes, i.e. $O(\Delta S^2)$. This procedure and the transformation back to the original option problem can increase the computational cost.

Therefore, the transformation appears to eliminate the problems connected with the approximation of the degenerate convection-diffusion Black-Scholes equation, but does not effectively solve those problems, especially for large values of r compared to σ^2.

3 Finite Difference Approach

Using a PDE approach as described in section 2, we have to decide which numerical scheme to adopt. For this reason, we first recall some results from numerical analysis of finite difference method and then examine in more detail how the numerical schemes usually adopted in finance can cope with the discontinuity of the initial condition.

3.1 A background analysis of numerical schemes

As usual, the S-domain is truncated at the cautelative value S_{max}. The computational domain $[0, S_{max}] \times [0, T]$ is discretized by a uniform mesh with steps ΔS, Δt in order to obtain nodes $(j\Delta S, n\Delta t)$, $j = 0, ..., M$, $n = 0, ..., N$ so that $S_M = S_{max} = M\Delta S$ and $T = N\Delta t$.

1. The choice of a specific numerical scheme is based on its property of convergence. Such a requirement is specified by the Lax's equivalence theorem: Given a properly posed linear initial-value problem and a linear finite difference approximation to it that satisfies the consistency condition, stability is the necessary and sufficient condition for convergence.

2. The stability turns out to be defined in terms of boundedness of the discrete solution at the fixed time T as Δt and ΔS tend to zero and when j or n tends to infinity. The practical consequence of this definition of stability is that a norm of the difference matrix A compatible with a vector norm must satisfy the condition $\|A\| < 1$ when the solution of the PDE does not increase as t increases, (Smith [23] page. 48).

3. Given a diagonally dominant matrix $A = [a_{ij}]$, let $\delta_i =| a_{ii} | - \sum_{j\neq i} | a_{ij} |$ then

$$\| A^{-1} \|_\infty < \frac{1}{\min_i \delta_i} \tag{8}$$

holds.

4. The parabolic nature of the (B-S) equation ensures that the solution obeys a maximum principle

$$\max_{S\in[0,S_{max}]} | V(t_1, S) | \geq \max_{S\in[0,S_{max}]} | V(t_2, S) | \qquad t_1 \leq t_2 \tag{9}$$

The numerical solution does not always satisfy a corresponding discrete version of the maximum principle, especially in the presence of boundary layers. If that condition is violated then the numerical solution may exhibit spurious wiggles near sharp gradients. As consequence, even though the numerical method converges, it often yields approximate solutions that differ qualitatively from corresponding exact solutions.

3.2 Explicit centered scheme

The Black-Scholes equation can be discretized by means of the two-level time explicit finite difference scheme, over a uniform mesh. The first derivative $\frac{\partial V}{\partial t}$ is discretized by means of a forward difference, while the terms $\frac{\partial V}{\partial S}$ and $\frac{\partial^2 V}{\partial S^2}$ by means of a centered difference. The approximations V_j^n of V at the grid points $(j\Delta S, n\Delta t)$ for every $j = 1, ..., M-1$ satisfy the finite difference scheme:

$$V^{n+1} = AV^n + b \tag{10}$$

where A is a tridiagonal matrix with

$$A = \text{tridiag}\left\{\Delta t\left[\frac{1}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2 - \frac{rS_j}{2\Delta S}\right], 1-\Delta t\left[r+\left(\frac{\sigma S_j}{\Delta S}\right)^2\right], \Delta t\left[\frac{1}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2 + \frac{rS_j}{2\Delta S}\right]\right\}, \tag{11}$$

$S_j = S(j\Delta S)$ and the vector b contains the boundary conditions (in the actual case $b \equiv 0$). The stability condition $\| A \|_\infty < 1$, entails that if $r/\sigma^2 < 1$ then:

$$\Delta t < \frac{1}{\frac{r}{2} + \left(\frac{\sigma S_M}{\Delta S}\right)^2}. \tag{12}$$

High accuracy demands for both small ΔS and high cautelative value of S_M. Thus stability requires prohibitively small values of Δt, unless S_M assumes a small value. Nothing can be said if $r/\sigma^2 > 1$. In this case the convection dominated behavior becomes more important.

3.3 Implicit scheme: a centered difference for $\frac{\partial V}{\partial S}$

We have the difference equation:

$$AV^{n+1} = V^n \tag{13}$$

where A is the tridiagonal matrix:

$$A = \text{tridiag}\left\{-\frac{\Delta t}{2}\left[\left(\frac{\sigma S_j}{\Delta S}\right)^2 - \frac{rS_j}{\Delta S}\right], 1+\Delta t\left[r+\left(\frac{\sigma S_j}{\Delta S}\right)^2\right], -\frac{\Delta t}{2}\left[\left(\frac{\sigma S_j}{\Delta S}\right)^2 + \frac{rS_j}{\Delta S}\right]\right\} \tag{14}$$

Under the restrictive hypothesis $r < \sigma^2$ which renders $(\frac{\sigma S_j}{\Delta S})^2 - \frac{rS_j}{\Delta S} > 0$ we have $\| A^{-1} \|_\infty \leq \frac{1}{1+r\Delta t}$.

From $\| V^{n+1} \|_\infty = \| A^{-1}V^n \|_\infty \leq \| A^{-1} \|_\infty \cdot \| V^n \|_\infty$ we have:

$$\max_j | V_j^{n+1} | \leq \| A^{-1} \|_\infty \max_j | V_j^n | \leq \frac{1}{1+r\Delta t} \max_j | V_j^n | \leq \max_j | V_j^n | . \tag{15}$$

Then the numerical solution satisfies a discrete version of the maximum principle and the numerical solution doesn't exhibit spurious wiggles near sharp concentration fronts.

In general, however, the centered differences are prone to introduce spurious oscillations if the condition $\sigma^2 > r$ is violated. In this case, it can be shown that some of eigenvalues of the iteration matrix can become complex and as consequence spurious oscillations are generated near the barriers. The oscillations are imputable to the fact that, for large ΔS, the numerical scheme is not able to cope with the steep gradient near the barrier. In Figures 3a-b the condition $\sigma^2 > r$ is violated and spurious wiggles arise which increase when ΔS increases (in Figure 3a-b we set respectively $\Delta S = 0.2$ and $\Delta S = 0.5$).

3.4 Implicit scheme: an upwind scheme for $\frac{\partial V}{\partial S}$

In order to eliminate undesired spurious wiggles, some specific numerical schemes, such as the upwind scheme, can be employed. Unfortunately, they can excessively smear the sharp front solution. This smearing mimics the effect of an enhanced diffusion coefficient, i.e., the physical diffusion is augmented by a numerical diffusion term, which vanishes as the grid step $\Delta S \to 0$, but artificially smears the numerical solution on any realistic spatial grid. The choice between spurious wiggles and numerical diffusion arises in most numerical methods for the advection-diffusion equation.

In the upwind scheme, we obtain the difference equation:

$$AV^{n+1} = V^n \tag{16}$$

where A is the tridiagonal matrix:

$$A = \text{tridiag}\left\{-\frac{\Delta t}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2, 1+\Delta t\left[r+\frac{rS_j}{\Delta S}+\left(\frac{\sigma S_j}{\Delta S}\right)^2\right], -\Delta t\left[\frac{1}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2+\frac{rS_j}{\Delta S}\right]\right\} \tag{17}$$

from which $\| A^{-1} \|_\infty \leq \frac{1}{1+r\Delta t}$, so that the numerical solution satisfies a discrete version of the maximum principle. However, the difference scheme smears the sharp front. We can interpret the smearing by examining the truncation error more closely. The upwind scheme introduces a local truncation error of the order $O(\Delta S)$. In fact, the upwind approximation $\frac{V_{j+1}^n - V_j^n}{\Delta S}$ of $\frac{\partial V}{\partial S}$ can be written as:

$$\frac{V_{j+1}^n - V_j^n}{2\Delta S} - \frac{\Delta S}{2}\frac{V_{j-1}^n - 2V_j^n + V_{j+1}^n}{\Delta S^2}, \tag{18}$$

namely it corresponds to a centered difference approximation of the regularized operator $\frac{\partial V}{\partial S} + \frac{\Delta S}{2}\frac{\partial^2 V}{\partial S^2}$. In other terms, it introduces a numerical dissipation which can be regarded as a direct discretization of the artificial viscous term $\frac{\Delta S}{2}\frac{\partial^2 V}{\partial S^2}$ and that makes the approximation only first order accurate. Therefore, up to terms that are $O(\Delta t, \Delta S^2)$ the scheme is an approximation of the PDE:

$$-\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + (\frac{1}{2}\sigma^2 S^2 + \frac{\Delta S}{2}rS)\frac{\partial^2 V}{\partial S^2} - rV = 0 \tag{19}$$

The numerical diffusion term $\frac{\Delta S}{2}rS$, which vanishes as $\Delta S \to 0$, artificially smears the numerical solution on any realistic spatial grid, particularly when the volatility is small,

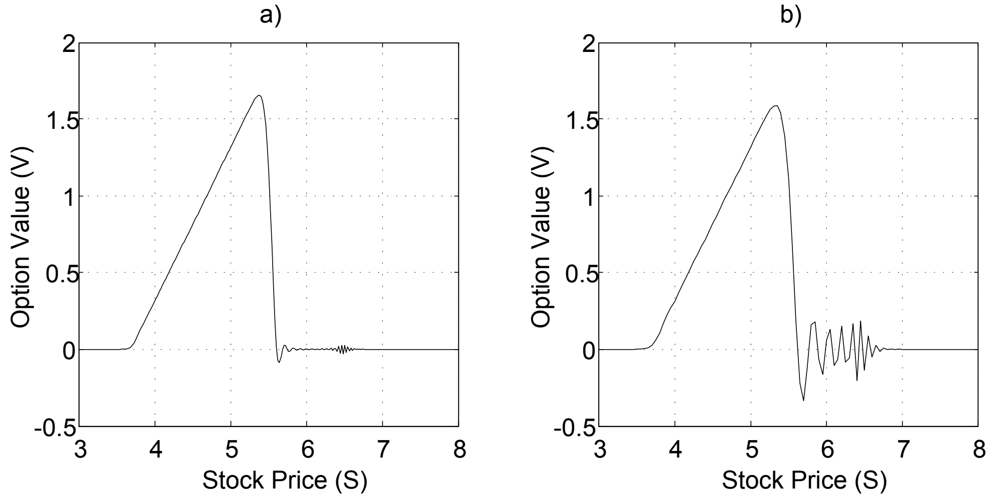

Figure 3: *Presence of spurious wiggles in the implicit centered scheme when the condition* $\sigma^2 > r$ *is violated. In figure 3a-b we set respectively* $\Delta S = 0.2$ *and* $\Delta S = 0.5$

compare Figure 4a. Such a numerical diffusion disappears when l, u assumes high values so that $\sigma^2 S^2/2 >> rS\Delta S/2$, compare Figure 4b.

3.5 Crank-Nicolson scheme

The analysis of Crank-Nicolson scheme on the backward B-S equation is difficult because of the advection term $\frac{\partial V}{\partial S}$. So a suitable coordinate transformation is used to eliminate troublesome terms of the equation. Putting:

$$S_1 = Se^{r(T-t)}, V_1 = Ve^{r(T-t)}, t_1 = \sigma^2(T-t) \tag{20}$$

the original backward B-S equation becomes

$$\frac{\partial V_1}{\partial t_1} = \frac{1}{2} S_1^2 \frac{\partial^2 V_1}{\partial S_1^2} \tag{21}$$

(from now on the index 1 will be dropped for simplicity of notation). The adopted transformation of coordinates changes a rectangular domain into a trapezoidal one. The original and new steps ΔS, ΔS_1 are related by $\Delta S = \Delta S_1 \exp(rT)$ at $t = T$. Crank-Nicolson scheme provides the difference equation:

$$AV^{n+1} = BV^n \tag{22}$$

with A, B tridiagonal matrices:

$$A = tridiag \left\{ -\left(\frac{S_j}{2\Delta S}\right)^2, \frac{1}{\Delta t} + \frac{1}{2}\left(\frac{S_j}{\Delta S}\right)^2, -\left(\frac{S_j}{2\Delta S}\right)^2 \right\}, \tag{23}$$

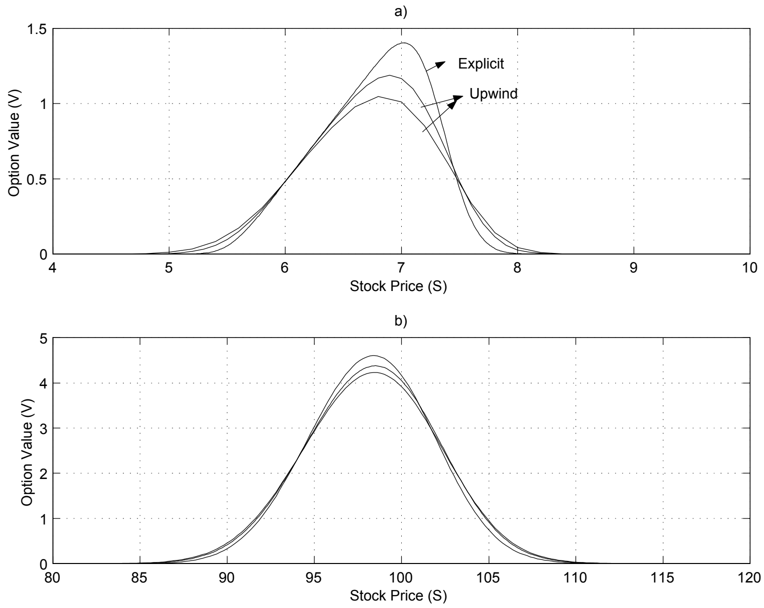

Figure 4: *Effect of the spurious diffusion term in the upwind method compared with a centered explicit scheme. In figure 4a barrier values at* $l = 4$, $u = 8$, $K = 6$. *In figure 4b barrier values at* $l = 90$, $u = 110$, $K = 100$. *The upwind scheme is solved respectively with* $\Delta S = 0.1$ *and* $\Delta S = 0.2$. *Other parameters:* $r = 1$, $\sigma = 0.1$, $\Delta t = 0.001$. *We plot the solution after the first monitoring date.*

$$B = \text{tridiag}\left\{ \left(\frac{S_j}{2\Delta S}\right)^2, \frac{1}{\Delta t} - \frac{1}{2}\left(\frac{S_j}{\Delta S}\right)^2, \left(\frac{S_j}{2\Delta S}\right)^2 \right\}. \tag{24}$$

Putting $A = \frac{1}{\Delta t}I + C$, $B = \frac{1}{\Delta t}I - C$, with C tridiagonal matrix we have $V^{n+1} = (I + \Delta t C)^{-1}(I - \Delta t C)V^n$. Then $D = (I + \Delta t C)^{-1}(I - \Delta t C)$ is the iteration matrix. The matrices $(I + \Delta t C)$ and $(I - \Delta t C)$ are symmetric and commutate. Then D is symmetric too and satisfies:

$$\| D \|_2 = \rho(D) = \max_s \left| \frac{1 - \Delta t \lambda_s(C)}{1 + \Delta t \lambda_s(C)} \right| < 1 \tag{25}$$

where $\lambda_s(C)$ denotes the s-th eigenvalue of C and $\rho(D)$ the spectral radius. Then the Crank-Nicolson scheme is unconditionally stable and consistent, so that, via Lax's equivalence theorem, is also convergent. From Gerschgorin theorem each eigenvalue $\lambda_s(C)$ of C belongs to the interval $\left[0, \left(\frac{S_s}{\Delta S}\right)^2\right]$, $s = 1, ..., M$. Then $\rho(C) := \max_s \lambda_s(C)$ satisfies $0 < \rho(C) < \left(\frac{S_M}{\Delta S}\right)^2$. When $\Delta S \to 0$ then $\rho(C) \to \infty$ and $\min_s \lambda_s(D) \to -1$, compare Figure 5a.

As a consequence, as well known, very slowly decaying finite oscillations can occur in the neighborhood of discontinuity in the initial values or near the barriers (compare Figure 6a).

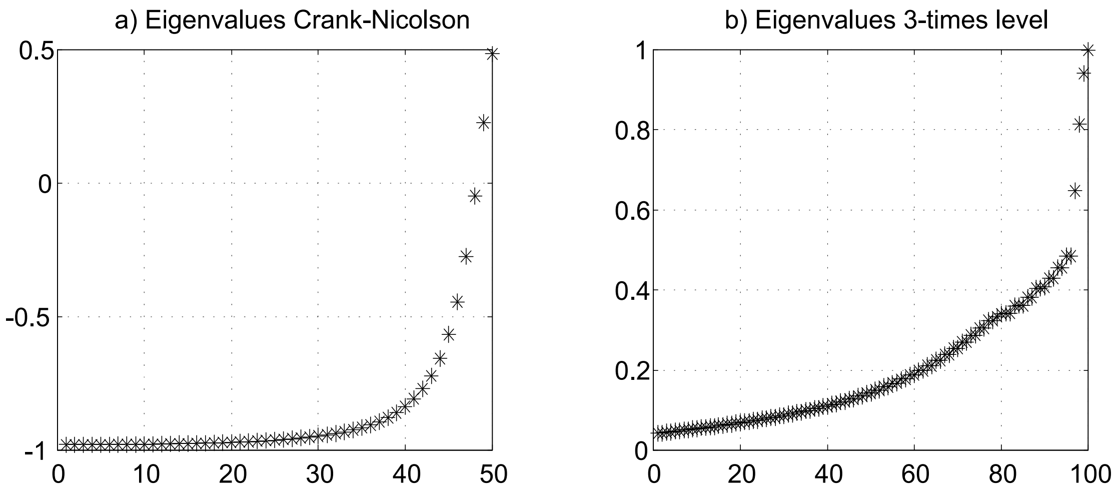

Figure 5: *a) Eigenvalues of the iteration matrix in the Crank-Nicolson scheme. b) Eigenvalues of the iteration matrix in the 3-level time scheme.*

The scheme would give a solution that is stable and free of unwanted oscillations if $0 < \lambda_s(D) < 1$, $s = 1, ..., M$. Such a condition is satisfied if $1 - \Delta t \lambda_s(C) > 0 \; \forall s$ and from $\lambda_s(C) < \rho(C) \leq \frac{1}{(\frac{S_M}{\Delta S})^2}$ we have:

$$\Delta t \leq \frac{1}{(\frac{S_M}{\Delta S})^2}. \tag{26}$$

Figure 6b shows how the oscillations shown in Figure 6a disappear when this time-step constraint is satisfied. However such a Δt value is prohibitively small.

3.6 Three-level time scheme

Various scheme have been proposed for treating problems introduced by considering discontinuous or non smooth boundary/initial conditions. These schemes aim to damp fast oscillations more effectively, by adjusting the spectrum of eigenvalues of the difference matrix. Three-(or more) time level schemes cure such problems quite dramatically.

The quantity $\frac{\partial V}{\partial t}$ is discretized with the three-level time scheme

$$\frac{3}{2}\frac{V_j^{n+1} - V_j^n}{\Delta t} + \frac{1}{2}\frac{V_j^n - V_j^{n-1}}{\Delta t} \tag{27}$$

whilst centered differences at $(n+1)\Delta t$ discretize $\frac{\partial V}{\partial S}$ and $\frac{\partial^2 V}{\partial S^2}$.

The difference equation is then given by:

$$AV^{n+1} = 4V^n - V^{n-1} \tag{28}$$

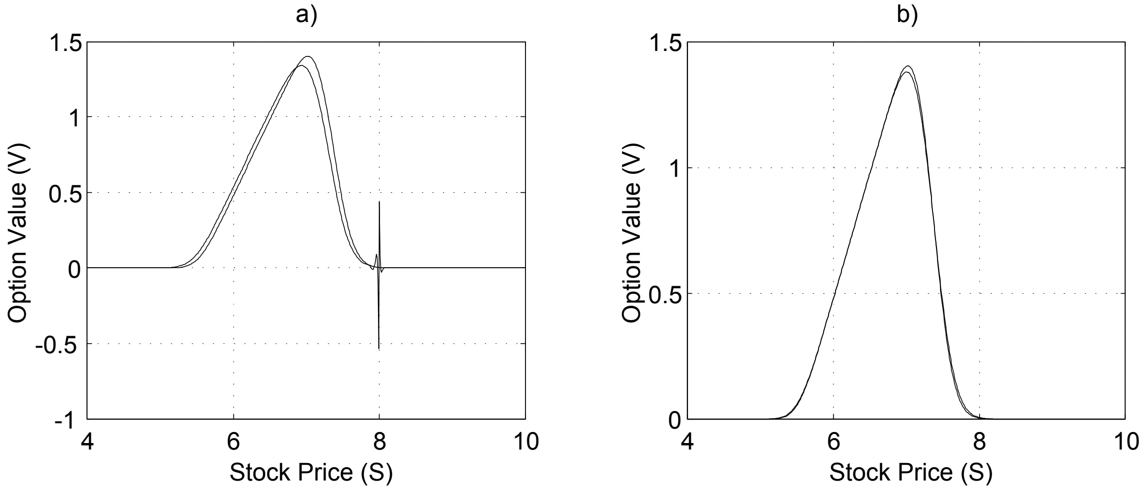

Figure 6: *a) Wiggles in the solution in the Crank-Nicolson scheme near the barrier when* $\Delta t = 0.1$*; b) same scheme when* $\Delta t = 10^{-6}$.

where:

$$A = \text{tridiag}\left\{-\Delta t\left[-\frac{rS_j}{\Delta S}+\left(\frac{\sigma S_j}{\Delta S}\right)^2\right], 3+2\Delta t\left[r+\left(\frac{\sigma S_j}{\Delta S}\right)^2\right], -\Delta t\left[\frac{rS_j}{\Delta S}+\left(\frac{\sigma S_j}{\Delta S}\right)^2\right]\right\}. \tag{29}$$

Such a scheme is second order accurate both in ΔS and in Δt.

Under the requirement $\left(\frac{\sigma S_j}{\Delta S}\right)^2 - \frac{rS_j}{\Delta S} > 0$ which leads to $\sigma^2 > r$, the matrix A is similar to a real symmetric tridiagonal matrix (Jacobi matrix), compare Smith [23] pag. 96. Then A admits M distinct eigenvalues and then M linear independent eigenvectors. From Gerschgorin theorem A is non singular and has eigenvalues $\lambda_j(A)$, $j = 1, ..., M$, which satisfy the inequality:

$$3 + 2r\Delta t < \lambda_j(A) < 3 + \Delta t\left[2r + 4\left(\frac{\sigma S_j}{\Delta S}\right)^2\right] \tag{30}$$

from which:

$$\frac{1}{3+\Delta t\left[2r+4\left(\frac{\sigma S_j}{\Delta S}\right)^2\right]} < \lambda_j(A^{-1}) < \frac{1}{3+2r\Delta t} \tag{31}$$

From (29) we have:

$$\left[\frac{V^{n+1}}{V^n}\right] = \left[\begin{array}{c|c} 4A^{-1} & -A^{-1} \\ I & \mathbf{0} \end{array}\right]\left[\frac{V^n}{V^{n-1}}\right] + \left[\frac{b}{\mathbf{0}}\right] \tag{32}$$

i.e., as $U_{n+1} = PU_n + d$, where $U_{n+1} = \begin{bmatrix} \frac{V^{n+1}}{V^n} \end{bmatrix}$. This technique has reduced a three-time level difference equation to a two-level one. The equations will be stable when each eigenvalue $\lambda_j(P)$ of P has modulus ≤ 1. The eigenvalues $\lambda_j(P)$ are the eigenvalues of the matrix:

$$\begin{bmatrix} 4\lambda_j(A^{-1}) & -\lambda_j(A^{-1}) \\ 1 & 0 \end{bmatrix} \tag{33}$$

Then $\lambda_j(P)$ satisfies the equation $f(\lambda_j(P)) = [\lambda_j(P)]^2 - \frac{4\lambda_j(P)}{\lambda_j(A)} + \frac{1}{\lambda_j(A)} = 0$, from which $f(0) = \frac{1}{\lambda_j(A)}$, $f(1) = 1 - \frac{3}{\lambda_j(A)} > 0$. When $\lambda_j(P) \in I\!R$ then $0 < \lambda_j(P) < 1$ holds. When $\lambda_j(P)$ is complex then $Re[\lambda_j(P)] = \frac{1}{\lambda_j(A)} > 0$ and $\mid \lambda_j(P) \mid = \frac{1}{\sqrt{\lambda_j(A)}} < 1$, compare Figure 5b. Therefore the equations, for a fixed mesh size, are unconditionally stable for all positive Δt and unaffected by spurious oscillations, being $\lambda_j(P) > 0$.

Unfortunately, the numerical analysis of this scheme is not able to say if the maximum principle is satisfied. However, having positive eigenvalues is a guarantee against oscillations respect to the use of the Crank-Nicolson scheme.

4 Finite Element approach

In this section we describe a different approach for solving problem (2) by means of the Galerkin Finite Element method [19], which is not common in Finance. In section 4.1 we briefly illustrate the standard Galerkin method in weighted Sobolev spaces; unfortunately, this method may present oscillations due to instability. In section 4.2 we analyse a generalized Galerkin method, which introduces a second order perturbation to the Black-Scholes equation, thus producing more stable numerical solutions.

4.1 Galerkin method

The *Galerkin Finite Element method* is a variational projection method, widely used in Physics and Engineering. The usefulness of Finite Elements in Finance has been recognized by several authors. However, to our knowledge the first to explore this approach in some details were [17], [26] Appendix D, [13], [14], [8], [9], [25] and [18]. With respect to Finite Difference methods, the Finite Element method can incorporate Neumann boundary conditions (which are often easier to obtain than Dirichlet conditions, when the price of the underlying goes to infinity) straightforwardly. It allows a more rigorous analysis of problems with discontinuous data; indeed, it is not based on the differential nature of the initial/boundary value problem and hence it can easily handle discontinuous solutions, not defined everywhere in the domain of integration. Moreover, it allows easily to introduce mesh adaption and to deal with irregular domains (arising, for instance, when knock-out barriers are imposed on a multiple-asset option) and, although it is not much used in Finance yet, it can in a natural way be used to approximate the variational inequalities encountered while studying American options. Finally, some of the measures of sensitivity to prices, such as the *greeks*, can be obtained more easily and precisely with Finite Elements, since the approximate solution is in piecewise polynomial form.

As mentioned above, the Finite Element method is not based on the strong pointwise differential formulation (2) of the knock-out barrier problem, but on a weaker integral formulation requiring less regularity for the unknown function V. This formulation is called *weak formulation* of problem (2).

First we remark that (2) has *perturbed ellipticity*, as its diffusion coefficient vanishes for $S \to 0$. Consequently, the appropriate spaces for studying problem (2) are the *weighted Sobolev spaces* ([15], p. 9).

Let us define the weighted inner product and norms

$$(\psi, \varphi) := \int_0^{S_{max}} \psi(x)\varphi(x)dx, \qquad |\psi| := \left(\int_0^{S_{max}} |\psi(x)|^2 dx\right)^{1/2} \tag{34}$$

$$\|\psi\| := \left(\int_0^{S_{max}} |\psi(x)|^2 dx + \int_0^{S_{max}} x^2 \left|\frac{d\psi}{dx}(x)\right|^2 dx\right)^{1/2}, \tag{35}$$

where the derivatives must be considered in a generalized sense (in the sense of distributions). Let us define the Hilbert space

$$\mathcal{H} := L^2(0, S_{max}) = \{\psi :]0, S_{max}[\to \mathbb{R}\ a.e. : |\psi| < +\infty\} \tag{36}$$

and the closure $\mathcal{V}$ of the space $C_0^\infty(]0, S_{max}[)$ in the Hilbert space

$$\mathcal{W} := \{\psi \in \mathcal{H} : \|\psi\| < +\infty\} \tag{37}$$

with respect to the norm $\|\cdot\|$. It can be proved that $\mathcal{V}$ is a separable Hilbert space ([15], p. 20), embedded in $\mathcal{H}$ with continuous and dense inclusion. Moreover $H_0^1(0, S_{max}) \subset \mathcal{V}$, with ([15], p. 47)

$$\|u\| \le c\|u\|_{H^1(0,S_{max})}, \quad \forall u \in H_0^1(0, S_{max}), \tag{38}$$

for some $c > 0$.

The variational formulation of (2), for $m = 0, \dots, F-1$, is the following

$$\text{find } V \in L^2(0, T; \mathcal{V}) \cap C^0([0, T]; \mathcal{H}) \text{ such that} \tag{39}$$

$$\frac{d}{dt}(V(t),\varphi)+\mathcal{A}(V(t),\varphi)=0, \qquad \forall\varphi\in\mathcal{V} \tag{40}$$

$$V(t_m)=V_{t_m}, \tag{41}$$

where we have set

$$\mathcal{A}(\psi,\varphi):=\frac{1}{2}\sigma^2\int_0^{S_{max}} S^2\frac{\partial\psi}{\partial S}\frac{\partial\varphi}{\partial S}dS+(\sigma^2-r)\int_0^{S_{max}} S\frac{\partial\psi}{\partial S}\varphi dS+r\int_0^{S_{max}}\psi\varphi dS \tag{42}$$

and

$$V_{t_0}:=(S-K)^+\mathbf{1}_{l,u}(S)\in\mathcal{H}, \tag{43}$$

$$V_{t_m}:=V(S,t_m)\mathbf{1}_{l,u}(S)\in\mathcal{H}, \tag{44}$$

It can be easily checked that the bilinear form $\mathcal{A}(\cdot,\cdot)$ is continuous and coercive on $\mathcal{V}$, i.e. there exist constants $\alpha,\gamma>0$ and $\lambda\in\mathbb{R}$ such that[2]

$$|\mathcal{A}(\psi,\varphi)|\le\gamma\|\psi\|\ \|\varphi\|, \qquad \forall\psi,\varphi\in\mathcal{V}, \tag{45}$$

$$\mathcal{A}(\psi,\psi)+\lambda|\psi|^2\ge\alpha\|\psi\|^2, \qquad \forall\psi\in\mathcal{V} \tag{46}$$

and hence problem (40)-(42) has a unique solution V ([3], p. 349).

Let us consider the family $\{\mathcal{V}_{\Delta S}\}_{\Delta S>0}$ of finite dimensional subspaces of $\mathcal{V}$ given by the piecewise linear functions over a uniform decomposition $S_0,S_1,\dots,S_{N_{\Delta S}}$ of the interval $[0,S_{max}]$, with $S_i=i\Delta S$, satisfying homogeneous boundary conditions. The general subspace $\mathcal{V}_{\Delta S}$ is included in $\mathcal{V}$, has finite dimension $N_{\Delta S}-1$ and approximates $\mathcal{V}$ as $\Delta S\to 0$ in a sense that will be specified below. The semidiscrete problem reads

$$\text{find } V_{\Delta S}:]t_m,t_{m+1}[\to\mathcal{V}_{\Delta S} \text{ such that} \tag{47}$$

$$\frac{d}{dt}(V_{\Delta S}(t),\varphi_{\Delta S})+\mathcal{A}(V_{\Delta S}(t),\varphi_{\Delta S})=0, \qquad \forall\varphi_{\Delta S}\in\mathcal{V}_{\Delta S}, \tag{48}$$

$$V_{\Delta S}(t_m)=V_{t_m,\Delta S} \tag{49}$$

where $V_{t_m,\Delta S}$ is a suitable approximation in $\mathcal{V}_{\Delta S}$ of the initial datum V_{t_m}, for instance the L^2-projection of V_{t_m} onto $\mathcal{V}_{\Delta S}$. Simple considerations show that problem (47)-(49) admits a unique solution $V_{\Delta S}$. Moreover, the density condition

$$\lim_{\Delta S\to 0}\inf_{\varphi_{\Delta S}\in\mathcal{V}_{\Delta S}}\|\varphi-\varphi_{\Delta S}\|=0, \quad \forall\varphi\in\mathcal{V} \tag{50}$$

[2] More exactly, (45) and (46) hold with $\lambda=0$ if $\sigma^2<3r/2$ and $\lambda=\sigma^2-3r/2+1$ if $\sigma^2\ge 3r/2$, $\alpha=\min(\sigma^2/2,3r/2-\sigma^2+\lambda)$ and $\gamma=\max(\sigma^2/2,r)+|\sigma^2-r|$.

holds, hence ([20], p. 160) Galerkin approximation converges to the weak solution V, i.e.

$$\lim_{\Delta S\to 0} V_{\Delta S} = V \text{ in } L^2(0,T;\mathcal{V})\cap C^0([0,T];\mathcal{H}). \tag{51}$$

If we set $V_{\Delta S}(S,t)=\sum_{j=1}^{N_{\Delta S}-1} V_j(t)\varphi_j(S)$, where the φ_j's are the so called *hat functions* generating $\mathcal{V}_{\Delta S}$, we get the following system of ordinary differential equations

$$\frac{d}{dt}\sum_{j=1}^{N_{\Delta S}-1}(\varphi_j,\varphi_i)V_j(t)+\sum_{j=1}^{N_{\Delta S}-1}\mathcal{A}(\varphi_j,\varphi_i)V_j(t)=0, \qquad i=1,\dots,N_{\Delta S}-1. \tag{52}$$

The integrals have been computed by means of the *iterated trapezoidal rule,* giving rise to the following differential system

$$\frac{d\mathbf{V}}{dt}(t)+A\mathbf{V}(t)=0, \quad t_m<t<t_{m+1} \tag{53}$$

$$\mathbf{V}(t_m)=\mathbf{V}_{t_m}, \tag{54}$$

where $A=tridiag(b_{i-1},a_i,c_i)$ is a square matrix of order $N_{\Delta S}-1$, with

$$b_{i-1}=-\frac{\sigma^2}{4\Delta S^2}\left[S_{i-1}^2+S_i^2\right]-\frac{\sigma^2-r}{2\Delta S}S_i; i=2,\dots,N_{\Delta S}-1, \tag{55}$$

$$a_i=\frac{\sigma^2}{4\Delta S^2}[S_{i-1}^2+2S_i^2+S_{i+1}^2]+r; i=1,\dots,N_{\Delta S}-1, \tag{56}$$

$$c_i=-\frac{\sigma^2}{4\Delta S^2}\left[S_i^2+S_{i+1}^2\right]+\frac{\sigma^2-r}{2\Delta S}S_i; i=1,\dots,N_{\Delta S}-2 \tag{57}$$

and $\mathbf{V}_{t_m}$ is the vector of the components of $V_{t_m,\Delta S}$ with respect to the basis $\{\varphi_j\}$. The reason why we do not perform exact integration is that the trapezoidal rule is accurate to the same order as our numerical scheme (i.e. $O(\Delta S^2)$) and hence it does not deteriorate accuracy. On the contrary, it simplifies computations and makes the numerical scheme more stable, as it corresponds to a *mass lumping* which concentrates the positive contributes due to the reaction term $r\int_0^{S_{max}} V\varphi dS$ on the diagonal of A.

After performing a uniform decomposition of the interval $[t_m,t_{m+1}]$, the *Backward Euler Method* yields the linear algebraic systems

$$(I+\Delta tA)\mathbf{V}^{n+1}=\mathbf{V}^n, \qquad n=0,\dots,N-1. \tag{58}$$

The stability analysis of this scheme leads to somewhat unusual conclusions. A sufficient stability condition is [16]

$$\|(I+\Delta tA)^{-1}\|_\infty<1. \tag{59}$$

There are two possible cases:

1. if $r \leq \sigma^2$ then the matrix $(I + \Delta t A)$ is an M-matrix and, using (8), one can easily show that

$$\|(I + \Delta t A)^{-1}\|_\infty \leq \frac{1}{1 + r\Delta t} < 1. \tag{60}$$

   In this way, all the V_j^n's are positive and the solution will not be affected by spurious oscillations, since any small error introduced in the computation will decay.

2. If $r > \sigma^2$ (convection-dominated case), then in order to have stability σ^2 should not be too small compared to r

$$\frac{2i^2 + 1}{2i - 2}\sigma^2 > r, \quad i = 2, \ldots, N_{\Delta S} - 1, \tag{61}$$

   otherwise condition (60) is no longer satisfied and stability is no longer guaranteed. We remark that condition (61) depends only on the option model and sometimes it can be violated for the first values of i if r is too large compared to σ^2, independently of how fine a grid spacing is used. However, this may not present a problem in practice since for small i the convective flux leaving node i is very small, because the velocity is only $-ri\Delta S$. Moreover, (60) is only a sufficient stability condition.

Nevertheless, in the presence of oscillations, a reduction of the discretization step ΔS can eventually eliminate instability, since, as ΔS tends to zero, the first nodes S_i move towards the origin, where the convective flux is small. Figure 7 shows the effect of a reduction of the step ΔS for an option with $\sigma^2 = 0.01$ and $r = 1$, where condition (61) is not satisfied for $i < 99$.

Problem (2) has been also approximated in time by means of *Crank-Nicolson method*, yielding

$$\left(I + \frac{\Delta t}{2}A\right)\mathbf{V}^{n+1} = \left(I - \frac{\Delta t}{2}A\right)\mathbf{V}^n, \qquad n = 0, \ldots, N - 1. \tag{62}$$

As reminded above for the finite difference case, this method is unconditionally stable and consistent, but it can produce slowly decaying oscillations where the exact solution has steep gradients due to the presence of a barrier. Figure 8 compares the approximations given by Backward Euler and Crank-Nicolson time steppings with the same discretization parameters.

4.2 Scharfetter-Gummel generalized Galerkin method

In Section 3.4, we have analysed the upwind scheme and we have shown how it introduces a numerical dissipation that can be regarded as a direct discretization of the artificial

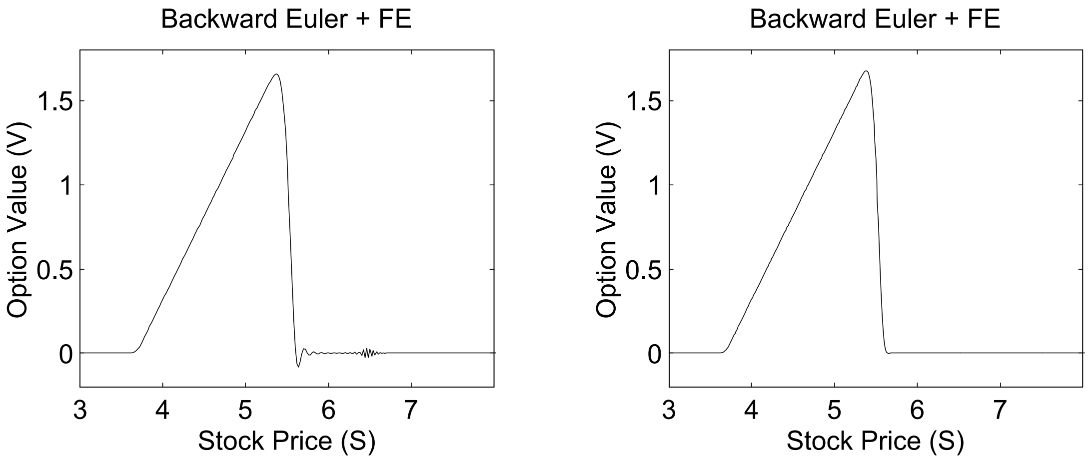

Figure 7: *Double barrier call option price at* $t_1 = 1/12\ y$ *to expiry when* $T = 1\ y$, $K = 4$, $\sigma^2 = 0.0001\ y^{-1}$, $r = 1.0\ y^{-1}$, $L = 2$, $U = 6$. *Backward Euler + Finite Element Galerkin method. Discretization parameters:* $\Delta t = 0.001$ *and* $\Delta S = 0.02$ *on the left,* $\Delta S = 0.01$ *on the right.*

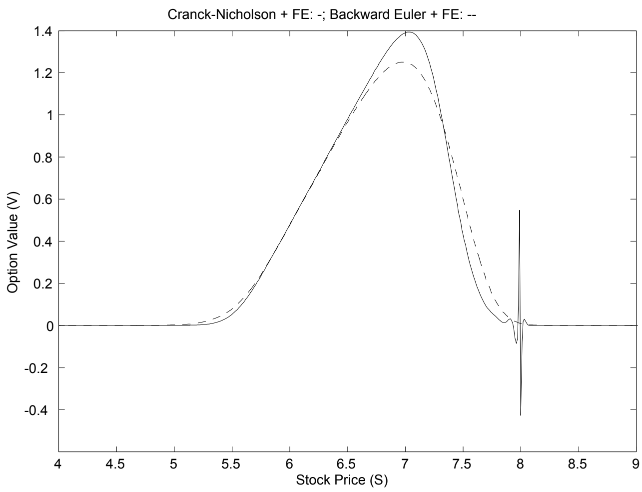

Figure 8: *Double barrier call option price at* $t_1 = 1/12\ y$ *to expiry when* $T = 1\ y$, $K = 6$, $\sigma^2 = 0.01\ y^{-1}$, $r = 1.0\ y^{-1}$, $L = 4$, $U = 8$. *Crank-Nicolson + Finite Element Galerkin method versus Backward Euler + Finite Element Galerkin method. Discretization parameters:* $\Delta t = 0.01$ *and* $\Delta S = 0.01$.

viscous term $-\frac{\sigma^2}{2}S^2\ Pe\frac{\partial^2 V}{\partial S^2}$, where

$$Pe := \frac{r\Delta S}{\sigma^2 S} \tag{63}$$

is the so called *local Péclet number*. This linear perturbation of (2) makes the upwind approximation only first order accurate in S. This interpretation of the concept of *upwinding* allows its extension to the Finite Element Method, where the notion of *non-centered* derivatives is not that obvious. In order to reduce the smearing effect of the artificial viscosity term, we can define the non-linear function

$$\phi(z) := \begin{cases} z - 1 + \frac{2z}{exp(2z)-1} & for\ z > 0 \\ 0 & for\ z = 0. \end{cases} \tag{64}$$

Let us consider the perturbed equation

$$\frac{\partial V}{\partial t} - \left(\frac{1}{2}\sigma^2 S^2 (1 + \phi(Pe)\right)\frac{\partial^2 V}{\partial S^2} - rS\frac{\partial V}{\partial S} + rV = 0. \tag{65}$$

We can easily see that the upwind perturbation corresponds to the choice $\phi(z) = z$.

Remark 4.1 *An asymptotic analysis of $\phi(z)$ as $z \to 0$ shows that*

$$\phi(z) = \frac{1}{3}z^2 + o(z^2). \tag{66}$$

Hence, we are introducing a second order perturbation in ΔS.

If we approximate the perturbed equation by means of Galerkin semidiscretization, we obtain the so called *Scharfetter-Gummel* scheme, which is stable and second order in ΔS

$$\text{find } V_{\Delta S} :]t_m, t_{m+1}[\to \mathcal{V}_{\Delta S} \text{ such that} \tag{67}$$

$$\frac{d}{dt}(V_{\Delta S}(t), \varphi_{\Delta S}) + \mathcal{A}_{\Delta S}(V_{\Delta S}(t), \varphi_{\Delta S}) = 0, \qquad \forall \varphi_{\Delta S} \in \mathcal{V}_{\Delta S}, \tag{68}$$

$$V_{\Delta S}(t_m) = V_{t_m, \Delta S}, \tag{69}$$

where

$$\mathcal{A}_{\Delta S}(V_{\Delta S}, \varphi_{\Delta S}) := \mathcal{A}(V_{\Delta S}, \varphi_{\Delta S}) - \frac{1}{2}\sigma^2 \int_0^{S_{max}} S^2 \frac{\partial^2 V_{\Delta S}}{\partial S^2}\ \varphi_{\Delta S}\ \phi(Pe(S))\ dS. \tag{70}$$

If we performed an integration by parts, we should compute contributes of the form

$$\frac{1}{2}\sigma^2 \int_{S_i}^{S_{i+1}} S^2 \frac{\partial V_{\Delta S}}{\partial S}\ \varphi_{\Delta S} \frac{d\phi(Pe(S))}{dS}\ dS. \tag{71}$$

Therefore, we replaced $\phi(z)$ by a piecewise constant function $\bar{\phi}_{\Delta S}$, so that

$$\begin{array}{l} \mathcal{A}_{\Delta S}(V_{\Delta S},\varphi_{\Delta S}) := \mathcal{A}(V_{\Delta S},\varphi_{\Delta S})+ \\ +\frac{1}{2}\sigma^2\sum_{i=0}^{N_{\Delta S}-1}\phi_{i+1/2}\left[\int_{S_i}^{S_{i+1}} S^2\frac{\partial V_{\Delta S}}{\partial S}\ \frac{\partial \varphi_{\Delta S}}{\partial S}\ dS + 2\int_{S_i}^{S_{i+1}} S\ \frac{\partial V_{\Delta S}}{\partial S}\ \varphi_{\Delta S}\ dS\right]+ \\ -\frac{1}{2}\sigma^2\sum_{i=0}^{N_{\Delta S}-1}\phi_{i+1/2}\left[S_{i+1}^2\ \frac{\partial V_{\Delta S}}{\partial S}\Big|_{i+1}^{-}\varphi_{\Delta S}(S_{i+1}) - S_i^2\ \frac{\partial V_{\Delta S}}{\partial S}\Big|_{i}^{+}\varphi_{\Delta S}(S_i)\right], \end{array} \tag{72}$$

where

$$\phi_{i+1/2} := \phi(Pe(S_{i+1/2}))$$

is the value of $\bar{\phi}_{\Delta S}$ in $]S_i, S_{i+1}]$ and $\frac{\partial V_{\Delta S}}{\partial S}\Big|_{i+1}^{-}$, $\frac{\partial V_{\Delta S}}{\partial S}\Big|_{i}^{+}$ are the interior traces of $\frac{\partial V_{\Delta S}}{\partial S}$ at the boundary of the interval $]S_i, S_{i+1}]$. We remark that $\lim_{\Delta S\to 0}\bar{\phi}_{\Delta S}(S) = 0$ for every $S > 0$, at a greater rate than for the upwind perturbation.

We get the following differential system

$$\frac{d\mathbf{V}}{dt}(t) + A_{\Delta S}\mathbf{V}(t) = 0, \quad t_m < t < t_{m+1} \tag{73}$$

$$\mathbf{V}(t_m) = \mathbf{V}_{t_m}, \tag{74}$$

where $A_{\Delta S} = tridiag(b_{i-1}^{\Delta S}, a_i^{\Delta S}, c_i^{\Delta S})$ is a square matrix of order $N_{\Delta S} - 1$, with

$$\begin{array}{l} b_{i-1}^{\Delta S} = b_{i-1} - \frac{\sigma^2}{4\Delta S^2}\phi_{i-1/2}\left[S_{i-1}^2 + S_i^2\right] - \frac{\sigma^2}{2\Delta S}\phi_{i-1/2}S_i + \frac{\sigma^2}{2\Delta S^2}\phi_{i-1/2}S_i^2, \\ \text{with } i = 2, \ldots, N_{\Delta S} - 1, \end{array} \tag{75}$$

$$\begin{array}{l} a_i^{\Delta S} = a_i + \frac{\sigma^2}{4\Delta S^2}[\phi_{i-1/2}S_{i-1}^2 - (\phi_{i-1/2} + \phi_{i+1/2})S_i^2 + \phi_{i+1/2}S_{i+1}^2]+ \\ +\frac{\sigma^2}{2\Delta S}(\phi_{i-1/2} - \phi_{i+1/2})S_i, \text{ with } i = 1, \ldots, N_{\Delta S} - 1, \end{array} \tag{76}$$

$$\begin{array}{l} c_i^{\Delta S} = c_i - \frac{\sigma^2}{4\Delta S^2}\phi_{i+1/2}\left[S_i^2 + S_{i+1}^2\right] + \frac{\sigma^2}{2\Delta S}\phi_{i+1/2}S_i + \frac{\sigma^2}{2\Delta S^2}\phi_{i+1/2}S_i^2, \\ \text{with } i = 1, \ldots, N_{\Delta S} - 2, \end{array} \tag{77}$$

Time discretization can be performed by means of both Euler scheme and Crank-Nicolson scheme. Figure 9 shows a detail of Scharfetter-Gummel approximation versus upwind approximation. It is evident that the exponential fitting reduces the smearing effect of the artificial diffusion method.

More sophisticated second order methods for dealing with convection-diffusion problems are ([19], pag. 268) the *strongly consistent stabilization methods for finite elements* such as SUPG, GALS or DWG, for which we refer the reader to the specialized literature.

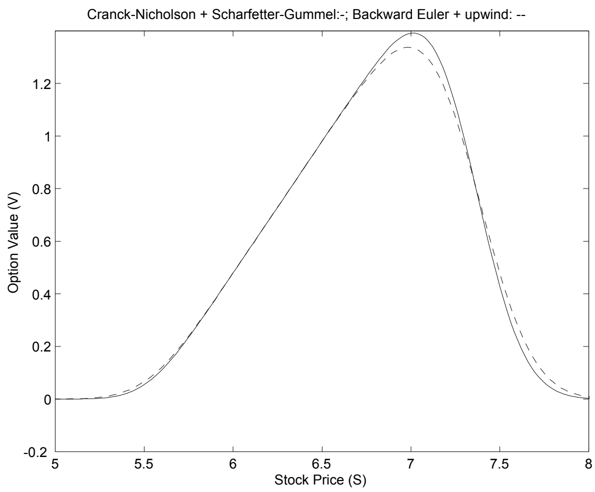

Figure 9: *Double barrier call option price at* $t_1 = 1/12$ *y to expiry when* $T = 1$ *y,* $K = 6$*,* $\sigma^2 = 0.01\ y^{-1}$*,* $r = 1.0\ y^{-1}$*,* $L = 4$*,* $U = 8$*. Crank-Nicolson + Scharfetter-Gummel Galerkin method versus Backward Euler + upwind difference method. Discretization parameters:* $\Delta t = 0.001$ *and* $\Delta S = 0.02$*.*

5 Concluding remarks

In this paper we have illustrated the problems encountered when we apply some common numerical scheme to the solution of PDE's arising in finance.

In particular we considered a non standard option pricing model and we studied how the effect of discontinuities in the initial/boundary conditions can combine with the inherent features of the Black-Scholes equation, i.e. the convective-diffusive flow with variable coefficients, and can deteriorate numerical approximation processes. The results of our study can be summarized in few points.

The best method to be used depends: a) on the value of the ratio r/σ^2, b) on the absolute values of the barriers and the spot price, although this condition can be removed by normalising the barriers and the spot price by the strike price, exploiting the homogeneity of the pricing function. From the carried out analysis we may conclude:

Transformations of the original Black-Scholes equation into diffusion problems are not robust, hence they are of no help for approximating convection-dominated problems. Moreover, such transformation are not always feasible for more complex option models. For instance, Asian option are modeled by partial differential equations with two space dimensions, which can be written in self-adjoint diffusion form only under suitable hypothesis on the velocity field [4].

The presence of discontinuous initial data imposes severe restrictions in the choice of the step ΔS, in order to obtain an accurate approximation of the rapid variations of V. Otherwise, a large approximation error occurs, which can propagate during time iteration. As a consequence, the choice of the numerical method must move towards those schemes having the iteration matrix characterized by a spectrum which allows a fast damping of errors of any kind.

In particular:

- Explicit methods are unsuitable, since they impose severe restrictions on the time step.
- When r is small compared to σ^2 (diffusion-dominated case), the remaining numerical methods are almost equivalent in terms of both accuracy and efficiency.
- For increasing r (convection-dominated case), it is convenient to discretize directly problem (2), eventually resorting to more accurate stabilization methods than the upwind scheme.

In conclusion, the parabolic (diffusive) nature of the problem can help in regularising the solution, although attention has to be devoted to the choice of the approximating method depending on the financial nature of the problem (contractual provisions) and the particular parameter values. Moreover, it cannot be always easy to find a simple expression for the stability constraints on the time step when we deal with more general processes, and so we should be careful in using naive implementations of numerical schemes.

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