# A Critique of the Crank Nicolson Scheme Strengths and Weaknesses for Financial Instrument Pricing

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## Summary

In this article we apply the Finite Difference Method (FDM) to the Black Scholes equation. In particular, we analyse the famous Crank Nicolson method that is very popular in financial engineering. Unfortunately, the method does not always produce accurate results and it is the objective of this article to enumerate the problems and then to propose more robust finite difference schemes. More detailed accounts of the current problem can be found in Duffy 2001 and Duffy 2004.

## 1 A short History of Crank Nicolson in Financial Engineering

The Crank Nicolson finite difference scheme was invented by John Crank and Phyllis Nicolson. They originally applied it to the heat equation and they approximated the solution of the heat equation on some finite grid by approximating the derivatives in space x and time t by finite differences. Much earlier, Richardson devised a finite difference scheme that was easy to compute but was numerically unstable and thus useless. The instability was not recognized until Crank, Nicolson and others carried out lengthy numerical calculations. In short, the Crank Nicolson method is numerically stable and it only requires the solution of a very simple system of linear equations (namely, a tridiagonal system) at every time level.

The Crank Nicolson method has become one of the most popular finite difference schemes for approximating the solution of the Black Scholes equation and its generalisations (see for example, Tavella 2000, Bhansali 1998). The method is essentially a second-order approximation to the time derivative that appears in the Black Scholes equation and this property, plus the fact that the method is stable and is easy to program makes it very appealing in practical applications. Numerous articles and publications in the financial engineering literature use Crank Nicolson as the de-facto scheme for time discretisation. Unfortunately, the method breaks down in certain situations and there are better and more robust alternatives that have been documented in the numerical analysis and computational fluid dynamics literature. To this end, we wish to discuss the shortcomings of the method and how they can be resolved.

## 2 What is Crank Nicolson, really?

The one-factor Black Scholes equation for a derivative quantity $V$ depending on an underlying $S$ is given by

$$-\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{1}$$

In general, this equation must be augmented by other boundary and initial conditions in order to ensure a unique solution. In some cases it may be possible to come up with an exact solution to this problem but in the most general cases we must resort to some kind of approximate method. In this article we discuss the Finite Difference Method and it is based on the tactic of replacing the continuous derivatives in (1) by *divided differences* defined on a discrete mesh (see Richtmyer 1967).

In order to motivate the Crank Nicolson scheme let us first consider the following *fully implicit scheme* that we define by replacing derivatives with respect to $S$ by three-point divided differences and the derivative with respect to $t$ by one-sided differences. The scheme is given by

$$\begin{aligned} &-\frac{V_j^{n+1}-V_j^n}{k}+rj\Delta S\left(\frac{V_{j+1}^{n+1}-V_{j-1}^{n+1}}{2\Delta S}\right) \\ &+\frac{1}{2}\sigma^2 j^2\Delta S^2\left(\frac{V_{j+1}^{n+1}-2V_j^{n+1}+V_{j-1}^{n+1}}{\Delta S^2}\right) \\ &= rV_j^{n+1} \end{aligned} \tag{2}$$

In general, the values of $V$ at time level $n$ are known and the values at time level $n+1$ need to be calculated. Rewriting (2) gives the new form

$$\begin{cases} a_j^{n+1}V_{j-1}^{n+1}+b_j^{n+1}V_j^{n+1}+c_j^{n+1}V_{j+1}^{n+1}=F_j^{n+1} \\ \text{where} \\ a_j^{n+1}=\left(\frac{1}{2}\sigma^2 j^2 k-\frac{krj}{2}\right) \\ b_j^{n+1}=-\left(1+\sigma^2 j^2 k+r\right) \\ c_j^{n+1}=\left(\frac{1}{2}\sigma^2 j^2 k+\frac{krj}{2}\right) \\ F_j^{n+1}=-V_j^n \end{cases} \tag{3}$$

This is a tridiagonal scheme that we solve at each time level using standard matrix solvers, for example LU decomposition (see Isaacson 1966, Duffy 2004). The fully implicit scheme has a number of desirable features. First, it is stable and there is no restriction on the relative sizes of the time mesh size $k$ and the space mesh size $\Delta S$. Furthermore, no spurious oscillations are to be seen in the solution or its $\Delta$ (as is the case with some other methods). A disadvantage is that it is only first order accurate in k. On the other hand, this can be rectified by using extrapolation and this results in a second-order scheme.

The Crank Nicolson is a variation of (2) but in this case we take averages of $V$ at levels $n$ and $n+1$ when approximating the derivative with respect to $t$. We define the quantity

$$V_j^{n+\frac{1}{2}}\equiv\frac{1}{2}\left(V_j^{n+1}+V_j^n\right) \tag{4}$$

Then the Crank Nicolson method is defined as follows:

$$\begin{aligned} &-\frac{V_j^{n+1}-V_j^n}{k}+rj\Delta S\left(\frac{V_{j+1}^{n+\frac{1}{2}}-V_{j-1}^{n+\frac{1}{2}}}{2\Delta S}\right) \\ &+\frac{1}{2}\sigma^2 j^2\Delta S^2\left(\frac{V_{j+1}^{n+\frac{1}{2}}-2V_j^{n+\frac{1}{2}}+V_{j-1}^{n+\frac{1}{2}}}{\Delta S^2}\right) \\ &= \tau V_j^{n+\frac{1}{2}} \end{aligned} \tag{5}$$

Again, this is a system that can be posed in the form (3) and hence can be solved by standard matrix solver techniques at each time level.

The Crank Nicolson method has gained wide acceptance in the financial literature and it seems to be the de-facto finite difference scheme for one-factor and two-factor Black Scholes equations. It has second order accuracy in the parameter k and is stable. Unfortunately, it has been known for some considerable time (Il'in 1969) that centred differencing schemes in space combined with averaging in time (what essentially CN is in this context) lead to spurious oscillations in the approximate solution. These oscillations have nothing to do with the physical or financial problem that the scheme is approximating.

# 3 The Problems with Crank Nicolson: the Details

We now give a detailed discussion of Crank Nicolson and when it breaks down or fails to live up to its perceived expectations.

## 3.1 A Critique of Crank-Nicolson

The Crank Nicolson method has become a very popular finite difference scheme for approximating the Black Scholes equation.

This equation is an example of a *convection-diffusion* equation and it has been known for some time that centred-difference schemes are inappropriate for approximating it (Il'in 1969, Duffy 1980). In fact, many independent discoveries of novel methods have been made in order to solve difficult convection-diffusion problems in fluid dynamics, atmospheric pollution modelling, semiconductor equations, the Fokker-Planck equation and groundwater transport (Morton 1996).

The main problem is that traditional finite difference schemes start to oscillate when the coefficient of the second derivative (the *diffusion* term) is very small or when the coefficient of the first derivative (the *convection* term) is large (or both). In this case, the mesh size h in the space direction must be smaller than a certain critical value if we wish to avoid these oscillations. This problem has been known since the 1950's (see de Allen 1955).

We now discuss the Crank Nicolson from a number of viewpoints. For convenience and generality reasons, we cast the Black Scholes equation as a generic parabolic initial boundary value problem in the domain $D = (A, B)X(0, T)$ where $A < B$:

$$
\begin{aligned}
&Lu \equiv -\frac{\partial u}{\partial t} + \sigma(x,t)\frac{\partial^2 u}{\partial x^2} + \mu(x,t)\frac{\partial u}{\partial x} + b(x,t)u = f(x,t) \ in\ D \\
&u(x,0) = \varphi(x),\ x \in (A,B) \\
&u(A,t) = g_0(t),\ u(B,t) = g_1(t),\ t \in (0,T)
\end{aligned}
\tag{6}
$$

In this case the time variable $t$ corresponds to increasing time while the space variable $x$ corresponds to the underlying asset price $S$. We specify Dirichlet boundary conditions on a finite space interval and this is a common situation for to several kinds of exotic options, for example barrier options. Actually, the system (6) is more general than the original Black Scholes equation.

## 3.2 How are Derivatives approximated?

There are two kinds of independent variables associated with the one-factor Black Scholes as can be seen in (6). These correspond to the $x$ and $t$ variables. We concentrate on the $x$ direction for the moment. We discretise in this direction using centred differences at the point $(jh, nk)$:

$$
\begin{cases}
\dfrac{\partial^2 u}{\partial x^2} \sim \dfrac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2} \\[2ex]
\dfrac{\partial u}{\partial x} \sim \dfrac{u_{j+1}^n - u_{j-1}^n}{2h}
\end{cases}
$$

Using this knowledge we can apply the Crank-Nicolson method to (6), namely:

$$
\begin{cases}
-\dfrac{u_j^{n+1} - u_j^n}{k} + \sigma_j^{n+\frac{1}{2}} \dfrac{u_{j+1}^{n+\frac{1}{2}} - 2u_j^{n+\frac{1}{2}} + u_{j-1}^{n+\frac{1}{2}}}{h^2} \\[2ex]
+ \mu_j^{n+\frac{1}{2}} \dfrac{u_{j+1}^{n+\frac{1}{2}} - u_{j-1}^{n+\frac{1}{2}}}{2h} \\[2ex]
+ b_j^{n+\frac{1}{2}} u_j^{n+\frac{1}{2}} = f_j^{n+\frac{1}{2}}
\end{cases}
\tag{7}
$$

A bit of simple arithmetic allows us to rewrite (7) in the standard form:

$$
\begin{cases}
a_j^n u_{j-1}^{n+1} + b_j^n u_j^{n+1} + c_j^n u_{j+1}^{n+1} = F_j^n \\
F_j^n \text{ known quantity}
\end{cases}
\tag{8}
$$

Of course, this system of equations can be posed in the form of a matrix system. A number of researchers have examined such systems in conjunction with convection-diffusion equations (for example, Farrell 2000, Morton 1996). A critical observation is that if the coefficient $a_j^n$ appearing in is not positive then the resulting solution will show oscillatory behaviour at best or produce non-physical solutions at worst.

This will give problems in general for Black Scholes applications where the volatility is a decaying function of time (see van Deventer 1997), for example:

$$\sigma(t) = \sigma_0 e^{-\alpha(T-t)}$$

where $\sigma_0$ and $\alpha$ are given constants.

We speak of a *singular perturbation* problem associated with problem (6) when the coefficient of the second derivative is small (see Duffy 1980). In this case traditional finite difference schemes perform badly at the *boundary layer* situated at $x = 0$. In fact, if we formally set volatility to zero in equation (7) we get a so-called *weakly stable* difference scheme (see Peaceman 1977) that approximate the *first-order hyperbolic equation*

$$-\frac{\partial u}{\partial t} + \mu\frac{\partial u}{\partial x} + bu = f$$

This has the consequence that the initial errors in the scheme are not dissipated and hence we can expect oscillations especially in the presence of rounding errors. We need other *one-sided schemes* in this degenerate case (Peaceman 1977, Duffy 1977).

## 3.3 Boundary Conditions

In general, we distinguish three kinds of boundary conditions:

- Dirichlet (as seen in the system (6))
- Neumann conditions
- Robin conditions

The last two boundary conditions involve the first derivative of the unknown u at the boundaries. We must then decide on how we are going to approximate this derivative. We can choose between first-order accurate one-sided schemes and *ghost points* (Thomas 1998) that produce a second-order approximation to the first derivative. We must thus be aware of the fact that the low-order accuracy at the boundary will adversely impact the second-order accuracy in the interior of the region of interest. To complicate matters, some models have a boundary condition involving the second derivative of u or even a 'linearity' boundary condition (see Tavella 2000).

Finally, the boundary conditions may be discontinuous. We may resort to nonuniform meshes to accommodate the discontinuities. This strategy will also destroy the second-order accuracy of the Crank-Nicolson method. The conclusion is that the wrong discrete boundary conditions adversely affect the accuracy of the finite difference scheme.

## 3.4 Initial Conditions

It is well-known that discontinuous initial conditions adversely impact the accuracy of finite difference schemes (see Smith 1978). In particular, the solution of the difference schemes exhibits oscillations just after t = 0 but the solution becomes more smooth as time goes on. This has consequences for options pricing applications because in general the initial condition (this is in fact a payoff function) is not always smooth. For example, the payoff function for a European call option is:

$$C = \max(S - K, 0)$$

where $K$ is the strike price and $S$ is the stock price. Its derivative is given by the jump function:

$$\frac{\partial C}{\partial S} = \begin{cases} 0, S \le K \\ 1, S > K \end{cases}$$

This derivative is discontinuous and in general we can expect to get bad accuracy at the points of discontinuity (in this case, at the strike price where at-the-money issues play an important role). It is possible to determine mathematically what the accuracy is in some special cases (Smith 1978) but numerical experiments show us that things are going wrong as well. Of course, if the option price is badly approximated there is not much hope of getting good approximations to the delta and gamma. This statement is borne out in practice. Another source of annoyance is that the boundary and initial conditions may not be *compatible* with each other. By compatibility, we mean that the solution is smooth at the corners (A, 0) and (B, 0) of the region of interest and we thus demand that the solution is the same irrespective of the direction from which we approach the corners. If we assume that $u(x, t)$ is continuous as we approach the boundaries, then we must satisfy the *compatibility conditions*:

$$\begin{cases} \varphi(A) \equiv u(A, 0) = g_0(0) \\ \varphi(B) \equiv u(B, 0) = g_1(0) \end{cases}$$

Failure to take these conditions into account in a finite difference scheme will lead to inaccuracies at the corner points of the region of interest. On the upside, the discontinuities are quickly damped out.

### 3.5 Proving Stability

Much of the literature uses the von Neumann theory to prove stability of finite difference schemes (Tavella 2000). This theory was developed by John von Neumann, a Hungarian-American mathematician, the father of the modern computer and probably one of the greatest brains of the twentieth century. Strictly speaking, the von Neumann approach is only valid for constant coefficient, linear initial value problems. The Black Scholes equation does not fall under this category. Furthermore, much work has been done in the engineering field to prove stability in other ways, for example using the maximum principle and matrix theory (Morton 1996, Duffy 1980). A discussion of von Neumann stability for the constant coefficient, linear convection-diffusion equation can be found in Thomas 1998.

## 4 An Introduction to Exponentially Fitted Finite Difference Schemes

### 4.1 A new Class of Robust Difference Schemes

Exponentially fitted schemes are stable, have good convergence properties and do not produce spurious oscillations. In order to motivate what an exponentially fitted difference scheme is, let us look at the simple boundary value problem:

$$\begin{aligned} &\sigma \frac{d^2u}{dx^2} + \mu \frac{du}{dx} = 0 \qquad in \qquad (A, B) \\ &u(A) = \beta_0,\ u(B) = \beta_1 \end{aligned} \tag{9}$$

Here we assume that $\sigma$ and $\mu$ are positive constants. We now approximate (9) by the difference scheme defined as follows:

$$\begin{aligned} &\sigma\rho D_+D_-U_j + \mu D_0 U_j = 0,\ j = 1, \ldots, J-1 \\ &U_0 = \beta_0,\ U_J = \beta_1. \end{aligned} \tag{10}$$

where $\rho$ is a so-called fitting factor (this factor is identically equal to 1 in the case of the centred difference scheme. We now choose $\rho$ so that the solutions of (9) and (10) are identical at the mesh-points. Some easy arithmetic shows that

$$\rho = \frac{\mu h}{2\sigma} \coth \frac{\mu h}{2\sigma}$$

where $\coth x$ is the hyperbolic cotangent function defined by

$$\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{e^{2x} + 1}{e^{2x} - 1}.$$

The fitting factor $\rho$ will be used when developing fitted difference schemes for variable coefficient problems. In particular, we discuss the following problem:

$$\begin{aligned} &\sigma(x)\frac{d^2u}{dx^2} + \mu(x)\frac{du}{dx} + b(x)u = f(x) \\ &u(A) = \beta_0,\ u(B) = \beta_1 \end{aligned} \tag{11}$$

where $\sigma$, $\mu$ and $b$ are given continuous functions, and

$$\sigma(x) \ge 0,\ \mu(x) \ge \alpha > 0,\ b(x) \le 0 \ \text{ for } \ x \in (A, B).$$

The fitted difference scheme that approximates (11) is defined by:

$$\begin{aligned} &\rho_j^h D_+D_-U_j + \mu_j D_0 U_j + b_j U_j = f_j, \quad j = 1, \ldots, J-1 \\ &U_0 = \beta_0, \quad U_J = \beta_1 \end{aligned} \tag{12}$$

where

$$\begin{aligned} &\rho_j^h = \frac{\mu_j h}{2} \coth \frac{\mu_j h}{2\sigma_j} \\ &\sigma_j = \sigma(x_j),\ \mu_j = \mu(x_j),\ b_j = b(x_j),\ f_j = f(x_j) \end{aligned} \tag{13}$$

We now state the following fundamental results (see Il'in 1969, Duffy 1980).

The solution of scheme (12) is *uniformly stable*, that is

$$|U_j| \le |\beta_0| + |\beta_1| + \frac{1}{\alpha} max_{k=1,\ldots,J}|f_k|, \quad j = 1, \ldots, J-1$$

>

Furthermore, scheme (12) is monotone in the sense that the matrix representation of (12)

$$AU = F$$

where $U = {}^t(U_1, \ldots, U_{J-1}), \quad F = {}^t(f_1, \ldots, f_{J-1})$ and

$$\mathbf{A} = \begin{pmatrix} \ddots & & \ddots & & 0 \\ & \ddots & & a_{j,j+1} & \\ \ddots & & a_{j,j} & & \ddots \\ & a_{j,j-1} & & \ddots & \\ 0 & & \ddots & & \ddots \end{pmatrix} \quad (14)$$

$$\begin{aligned} a_{j,j-1} &= \frac{\rho_j^h}{h^2} - \frac{\mu_j}{2h} > 0 \qquad \textit{always} \\ a_{j,j} &= -\frac{2\rho_j^h}{h^2} + b_j < 0 \qquad \textit{always} \\ a_{j,j+1} &= \frac{\rho_{j^h}}{h^2} + \frac{\mu_j}{2h} > 0 \qquad \textit{always} \end{aligned}$$

produces positive solutions from positive input.

Sufficient conditions for a difference scheme to be monotone have been investigated by many authors in the last 30 years; we mention the work of Samarski 1976 and Stoyan 1979.

Stoyan also produced stable and convergent difference schemes for the convection-diffusion equation producing results and conclusions that are similar to the author's work (see Duffy 1980).

Let $u$ and $U$ be the solutions of (11) and (12), respectively. Then

$$|u(x_j) - U_j| \le Mh$$

where $M$ is a positive constant that is independent of $h$ and $\sigma$ (Il'in (1969).

The conclusion is that the fitted scheme (12) is stable, convergent and produces no oscillations. In particular, the scheme 'degrades gracefully' to a well-known stable schemes when $\sigma$ tends to zero.

# 5 Exponentially Fitted Schemes for the Black Scholes Equation

We discretise the rectangle $[A, B] \times [0, T]$ as follows:

$$\begin{aligned} &A = x_0 < x_1 < \ldots < x_J = B \ (h = x_j - x_{j-1}), h \textit{ constant} \\ &0 = t_0 < t_1 < \ldots < t_N = T \ (k = T/N), k \textit{ constant} \end{aligned}$$

Consider again the operator $L$ in equation (6) defined by

$$Lu \equiv -\frac{\partial u}{\partial t} + \sigma(x,t)\frac{\partial^2 u}{\partial x^2} + \mu(x,t)\frac{\partial u}{\partial x} + b(x,t)u.$$

We replace the derivatives in this operator by their corresponding divided differences and we define the fitted operator $L_k^h$ by

$$L_k^h U_j^n \equiv -\frac{U_j^{n+1} - U_j^n}{k} + \rho_j^{n+1} D_+ D_- U_j^{n+1} + \mu_j^{n+1} D_0 U_j^{n+1} + b_j^{n+1} U_j^{n+1} \quad (15)$$

Here we use the notation

$$\varphi_j^{n+1} = \varphi(x_j, t_{n+1}) \text{ in general}$$

and

$$\rho_j^{n+1} \equiv \frac{\mu_j^{n+1} h}{2} \coth \frac{\mu_j^{n+1} h}{2\sigma_j^{n+1}}$$

We now formulate the fully-discrete scheme that approximates the initial boundary value problem (6):

Find a discrete function $\{U_j^n\}$ such that

$$\begin{aligned} &L_k^h U_j^n = f_j^{n+1}, \ j = 1, \ldots, J-1, \ n = 0, \ldots, N-1 \\ &U_0^n = g_0(t_n), \ U_J^n = g_1(t_n), \ n = 0, \ldots, N \\ &U_j^0 = \varphi(x_j), \ j = 1, \ldots, J-1 \end{aligned} \quad (16)$$

This is a two-level implicit scheme. We wish to prove that scheme (16) is stable and is consistent with the initial boundary value problem (6). We prove stability of (16) by the so-called discrete maximum principle instead of the von Neumann stability analysis. The von Neumann approach is well known but the discrete maximum principle is more general and easier to understand and to apply in practice. It is also the defacto standard technique for proving stability of finite difference and finite element schemes (see Morton 1996, Farrell 2000).

**Lemma 1** *Let the discrete function $w_j^n$ satisfy $L_k^h w_j^n \le 0$ in the interior of the mesh with $w_j^n \ge 0$ on the boundary $\Gamma$.*
*Then $w_j^n \ge 0, \forall j = 0, \ldots, J; n = 0, \ldots, N$.*

**Proof**: We transform the inequality $L_k^h w_j^n \le 0$ into an equivalent vector inequality. To this end, define the vector $W^n = {}^t(w_1^n, \ldots, w_{J-1}^n)$. Then the inequality $L_k^h w_j^n \le 0$ is equivalent to the vector inequality

$$A^n W^{n+1} \ge W^n \quad (17)$$

where

$$A^n = \begin{pmatrix} \ddots & & \ddots & & 0 \\ & \ddots & & t_j^n & \\ \ddots & & s_j^n & & \ddots \\ & r_j^n & & \ddots & \\ 0 & & \ddots & & \ddots \end{pmatrix}$$

$$r_j^n = \Big(-\frac{\rho_j^n}{h^2} + \frac{\mu_j^n}{2h}\Big)k$$
$$s_j^n = \Big(\frac{2\rho_j^n}{h^2} - b_j^n + k^{-1}\Big)k$$
$$t_j^n = \Big(-\Big(\frac{\rho_j^n}{h^2} + \frac{\mu_j^n}{2h}\Big)\Big)k$$

It is easy to show that the matrix $A^n$ has non-positive off-diagonal elements, has strictly positive diagonal elements and is irreducibly diagonally dominant. Hence (see Varga 1962 pages 84-85) $A^n$ is non-singular and its inverse is positive:

$$(A^n)^{-1} \geq 0$$

Using this result in (17) gives the desired result.

**Lemma 2** *Let $\{U_j^n\}$ be the solution of scheme (16) and suppose that*

$$max|U_j^n| \leq m \text{ on } \Gamma \text{ for all } j \text{ and } n$$
$$max|f_j^n| \leq N \text{ in } D \text{ for all } j \text{ and } n$$

*Then*

$$max_j|U_j^n| \leq -\frac{N}{\beta} + m \text{ in } \bar{D}$$

**Proof:** Define the discrete barrier function

$$w_j^n = -\frac{N}{\beta} + m \pm U_j^n$$

Then $w_j^n \geq 0$ on $\Gamma$. Furthermore,

$$L_k^h w_j^n \leq 0$$

Hence $w_j^n \geq 0$ in $\bar{Q}$ which proves the result.

Let $u(x,t)$ and $\{U_j^n\}$ be the solutions of (6) and (16), respectively.

Then

$$|u(x_j, t_n) - U_j^n| \leq M(h+k) \tag{18}$$

where $M$ is a constant that is independent of $h$, $k$ and $\sigma$.

This result shows that convergence is assured regardless of the size of $\sigma$. No classical scheme (for example, centred differencing in $x$ and Crank Nicolson in time) have error bounds of the form (18) where $M$ is independent of $h$, $k$ and $\sigma$.

Summarising, the advantages of the fitted scheme are:

- It is uniformly stable for all values of $h$, $k$ and $\sigma$.
- It is oscillation-free. Its solution converges to the exact solution of (6). In particular, it is a powerful scheme for the Black-Scholes equation and its generalisations.
- It is easily programmed, especially if we use object-oriented design and implementation techniques.

## 6 Problems with Small Volatility

We now examine some 'extreme' cases in system (16). In particular, we examine the cases

(pure convection/drift) $\sigma \to 0$
(pure diffusion/volatility) $\mu \to 0$

We shall see that the 'limiting' difference schemes are well-known schemes and this is reassuring. To examine the first extreme case we must know what the limiting properties of the hyperbolic cotangent function are:

$$\lim_{\sigma\to 0} \rho_j^n = \lim_{\sigma \to 0} \frac{\mu_j^n h}{2} \coth \frac{\mu_j^n h}{2\sigma_j^n}$$

We use the formula

$$\lim_{\sigma \to 0} \frac{\mu h}{2} \coth \frac{\mu h}{2\sigma} = \begin{cases} +\frac{\mu h}{2} & \text{if } \mu > 0 \\ -\frac{\mu h}{2} & \text{if } \mu < 0 \end{cases}$$

Inserting this result into the first equation in (16) gives us the first-order scheme

$$\mu > 0, \quad -\frac{U_j^{n+1} - U_j^n}{k} + \mu_j^{n+1}\frac{(U_{j+1}^{n+1} - U_j^{n+1})}{h} + b_j^{n+1}U_j^{n+1} = f_j^{n+1}$$
$$\mu < 0, \quad -\frac{U_j^{n+1} - U_j^n}{k} + \mu_j^{n+1}\frac{(U_j^{n+1} - U_{j-1}^{n+1})}{h} + b_j^{n+1}U_j^{n+1} = f_j^{n+1}$$

These are so-called implicit upwind schemes and are stable and convergent (Duffy 1977, Dautray 1993). We thus conclude that the fitted scheme degrades to an acceptable scheme in the limit. The case $\mu \to 0$ uses the formula

$$\lim_{x \to 0} x \coth x = 1$$

Then the first equation in system (16) reduces to the equation

$$-\frac{U_j^{n+1} - U_j^n}{k} + \sigma_j^{n+1} D_+ D_- U_j^{n+1} + b_j^{n+1} U_j^{n+1} = f_j^{n+1}$$

This is a standard approximation to pure diffusion problems and such schemes can be found in standard numerical analysis textbooks.

These limiting cases reassure us that the fitted method behaves well for 'extreme' parameter values.

## 7 Exponential Fitting and Exotic Options

We have applied the method to a range of plain and exotic European and American type options. In particular, we have applied it to various kinds of barrier options (see Topper 1998, Haug 1998), for example:

- Double barrier call options
- Single barrier call options
- Equations with time-dependent volatilities (for example, a linear function of time)
- Asymmetric plain vanilla power call options
- Asymmetric capped power call options

We have compared our results with those in Haug 1998 and Topper 1998 and they compare favourably (Mirani 2002). The main difference between these types lies in the specific payoff functions (initial conditions) and boundary conditions. Since we are working with a specific kind of parabolic problem these functions must be specified by us. For example, for a double barrier option we must give the value of the option at these barriers while for a single barrier option we define the 'down' barrier at $S = 0$. Summarising, the exponentially fitted finite difference scheme gives good approximations to the option price and delta of the above exotic option types. We have compared the results with Monte Carlo, Haug 1998 and Topper 1998.

# 8 Uniform Approximation of the Greeks

It is well known by now that CN produces bad approximation to option delta and gamma (see for example, Zvan 1997, Cooney 1999). Thus, we need to devise schemes that do give uniform approximation to option sensitivities, especially in the vicinity of the strike price K. The exponentially fitted scheme (16) is a good candidate and more information can be found in Duffy 2001 and Cooney 1999.

## 8.1 Is there more Hope? The Keller Scheme

In this section however, we give a short overview of the Box Scheme (Keller 1971) that resolves many of the problems associated with Crank Nicolson. In short, we reduce the second-order Black Scholes equation to a system of first-order equations containing at most first-order derivatives. We then approximate the first derivatives in *x and t* by averaging in a box. We motivate the box scheme by examining the generic parabolic initial boundary value problem in the space interval (0,1):

$$\begin{aligned}
&\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right) + cu + S,\ 0 < x < 1,\ t > 0\\
&u(x,0) = g(x),\ 0 < x < 1\\
&\alpha_0 u(0,t) + \alpha_1 a(0,t) u_x(0,t) = g_0(t)\\
&\beta_0 u(1,t) + \beta_1 a(1,t) u_x(1,t) = g_1(t)
\end{aligned} \tag{19}$$

Here $u$ is the (unknown) solution to the problem that satisfies the *self-adjoint equation* in (19) and it must also satisfy the initial and boundary conditions (note the latter contain derivatives of the unknown at the boundaries of the interval).In general, the coefficients in (19) are functions of both $x$ and $t$.

We now transform (19) to a first order system by defining a new variable $v$. The new transformed set of equations is given by:

$$\begin{aligned}
&a\frac{\partial u}{\partial x} = v\\
&\frac{\partial v}{\partial x} = \frac{\partial u}{\partial t} - cu - S\\
&u(x,0) = g(x)\\
&\alpha_0 u(0,t) + \alpha_1 v(0,t) = g_0(t)\\
&\beta_0 u(1,t) + \beta_1 v(1,t) = g_1(t)
\end{aligned} \tag{20}$$

We now see that we have to do with a first order system of equations with no derivatives on the boundaries!

We now need to introduce some notation. First, we define average values for $x$ and $t$ coordinates as follows:

$$\begin{aligned}
x_{j\pm 1/2} &= \tfrac{1}{2}(x_j + x_{j\pm 1})\\
t_{n\pm 1/2} &= \tfrac{1}{2}(t_n + t_{n\pm 1})
\end{aligned}$$

and for general nets (in principle the approximations to $u$ and $v$) by

$$\begin{aligned}
\phi^n_{j\pm 1/2} &= \tfrac{1}{2}(\phi^n_j + \phi^n_{j\pm 1})\\
\phi^{n\pm 1/2}_j &= \tfrac{1}{2}(\phi^n_j + \phi^{n\pm 1}_j)
\end{aligned}$$

Finally, we define notation for divided differences in the $x$ and $t$ directions as follows:

$$\begin{aligned}
D^-_x \phi^n_j &= h_j^{-1}(\phi^n_j - \phi^n_{j-1})\\
D^-_t \phi^n_j &= k_n^{-1}(\phi^n_j - \phi^{n-1}_j)
\end{aligned}$$

We are now ready for the new scheme. To this end, we use one-sided difference schemes in both directions while taking averages and we thus solve for both $u$ and $v$ simultaneously at each time level:

$$\begin{aligned}
&a^n_{j-1/2} D^-_x u^n_j = v^n_{j-1/2}\\
&D^-_x v^{n-1/2}_j = D^-_t u^n_{j-1/2} - c^{n-1/2}_{j-1/2} u^{n-1/2}_{j-1/2} - S^{n-1/2}_{j-1/2}\\
&1 \le j \le J,\ 1 \le n \le N
\end{aligned} \tag{21}$$

The corresponding boundary and initial conditions are:

$$\left.\begin{aligned}
\alpha_0 u^n_0 + \alpha_1 v^n_0 &= g^n_0\\
\beta_0 u^n_J + \beta_1 v^n_J &= g^n
\end{aligned}\right\} 1 \le n \le N \tag{22}$$

The box scheme has a number of very desirable properties, namely:

a) it is simple, efficient and easy to program b) it is unconditionally stable c) it approximates u and its partial derivative in $x$ with second-order accuracy. For the Black Scholes equation this means that we can approximate both option price and the option delta without trace of spurious oscillation as is experienced with Crank Nicolson d) Richardson extrapolation is applicable and yields two orders of accuracy improvement per extrapolation (with nonuniform nets!) e) It supports data, coefficients and solutions that are only piecewise smooth. In a financial setting it is able to model piecewise smooth payoff functions. We then define the approximate initial condition as follows:

$$v^0_{j-\frac{1}{2}} = a^0_{j-\frac{1}{2}} \frac{dg\left(x_{j-1/2}\right)}{dx}, 1 \le j \le J \tag{23}$$

For piecewise smooth boundary conditions we use the following tactic:

$$\begin{aligned} &\alpha_0 u_0^{n-\frac{1}{2}} + \alpha_1 v_0^{n-\frac{1}{2}} = g_0^{n-\frac{1}{2}} \\ &\beta_0 u_J^{n-\frac{1}{2}} + \beta_1 v_J^{n-\frac{1}{2}} = g_1^{n-\frac{1}{2}} \\ &1 \le n \le N \\ &\textit{Discontinuities at } t = t_n! \end{aligned} \tag{24}$$

Of course we are assuming that the mesh points are sitting on the discontinuities! f)

## 9 Conclusions

We have discussed the popular Crank Nicolson method from a number of viewpoints. In particular, we have made an inventory of the situations where it breaks down or where it deviates from our expectations:

- The standard von Neumann stability analysis fail to predict the infamous spurious oscillation problem. Hedging applications that use CN will run the risk of inaccuracy at values in the pay off function where this function is not smooth (for example, the strike price)
- Second-order accuracy is lost when using non-uniform meshes. Sometimes uniform meshes are not sufficient to approximate the exact solution in a boundary layer (small volatility) or with nasty pay-off functions (for example, binary options or barriers options with discrete and intermittent barriers). A good discussion of how Crank Nicolson breaks down for barrier options is given in Tavella 2000.
- There are finite difference schemes that are just as good as, or even better than Crank Nicolson, for example fully implicit schemes with extrapolation or Runge-Kutta (Crouzeix 1975).
- For two-factor and multi-factor problems, we use predictor-corrector, Alternating Direction Implicit (ADI) and Operator Splitting methods (see Peaceman 1977, Janenko 1971, Sun 1999). In these cases we see that Crank Nicolson is just one possibility for time discretisation.

A modest proposal would be to investigate robust and effective alternatives to the Crank Nicolson schemes. This will hopefully improve the *FDM gene pool* as it were.

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