PART FOUR

PROBLEMS OF COMPUTATION

CHAPTER XXI

MAXIMIZATION OF A LINEAR FUNCTION OF VARIABLES SUBJECT TO LINEAR INEQUALITIES[1]

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The general problem indicated in the title is easily transformed, by any one of several methods, to one which maximizes a linear form of nonnegative variables subject to a system of linear equalities. For example, consider the linear inequality $ax + by + c > 0$. The linear inequality can be replaced by a linear equality in nonnegative variables by writing, instead, $a(x_1 - x_2) + b(y_1 - y_2) + c - z = 0$, where $x_1 \geqq 0$, $x_2 \geqq 0$, $y_1 \geqq 0$, $y_2 \geqq 0$, $z \geqq 0$. The basic problem throughout this chapter will be considered in the following form:

PROBLEM: *Find the values of* $\lambda_1, \lambda_2, \cdots, \lambda_n$ *which maximize the linear form*

$$\lambda_1 c_1 + \lambda_2 c_2 + \cdots + \lambda_n c_n \tag{1}$$

subject to the conditions that

$$\lambda_j \geqq 0 \qquad (j = 1, 2, \cdots, n) \tag{2}$$

and

$$\begin{aligned} \lambda_1 a_{11} + \lambda_2 a_{12} + \cdots + \lambda_n a_{1n} &= b_1, \\ \lambda_1 a_{21} + \lambda_2 a_{22} + \cdots + \lambda_n a_{2n} &= b_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ \lambda_1 a_{m1} + \lambda_2 a_{m2} + \cdots + \lambda_n a_{mn} &= b_m, \end{aligned} \tag{3}$$

where a_{ij}, b_i, c_j *are constants* $(i = 1, 2, \cdots, m; j = 1, 2, \cdots, n)$.

[1] The author wishes to acknowledge that his work on this subject stemmed from discussions in the spring of 1947 with Marshall K. Wood, in connection with Air Force programming methods. The general nature of the "simplex" approach (as the method discussed here is known) was stimulated by discussions with Leonid Hurwicz.

The author is indebted to T. C. Koopmans, whose constructive observations regarding properties of the simplex led directly to a proof of the method in the early fall of 1947. Emil D. Schell assisted in the preparation of various versions of this chapter. Jack Laderman has written a set of detailed working instructions and has tested this and other proposed techniques on several examples.

Each column of coefficients in (3) may be viewed as representing the coordinates of a point in Euclidean R_m space. Let P_j denote the jth column of coefficients and P_0 the constants on the right-hand side, i.e., by definition,

$$(4)\qquad [P_1, P_2, \cdots, P_n; P_0] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}.$$

The basic problem then is to determine nonnegative $\lambda_j \geqq 0$ such that

$$(5)\qquad \lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_n P_n = P_0,$$

$$(6)\qquad \lambda_1 c_1 + \lambda_2 c_2 + \cdots + \lambda_n c_n = z = \max.$$

A set of λ_j which satisfy (5) without necessarily yielding the maximum in (6) will be termed a *feasible* solution; one which maximizes (6) will be called a *maximum feasible* solution. The purpose of this chapter is to discuss the so-called "simplex" technique, which consists in constructing first a feasible, and then a maximum feasible, solution. In many applications, of course, feasible solutions are easily obtained by inspection. For this reason, and because an arbitrary feasible solution can be obtained in a manner analogous to the construction of a maximum feasible solution, we shall consider first the construction of a maximum feasible solution from a given feasible solution.[2]

ASSUMPTION (nondegeneracy): *Every subset of m points from the set* $(P_0; P_1, P_2, \cdots, P_n)$ *is linearly independent.*

The theorems given in Sections 1 and 2 below come about naturally in the construction of a feasible and a maximum feasible solution to (5)

[2] The nondegeneracy assumption has been made to simplify the development that follows. There are obvious ways in which this assumption could be weakened. For example, the m equations implied in (5) may not all be linearly independent, in which case $k < m$ independent equations could be chosen and the remainder dropped. When this is done it may still be true that P_0 is linearly dependent on less than k of the P_i. One way to avoid this type of "degeneracy" is to alter slightly the values of the components of P_0. This method is extensively employed in the transportation problem [XXIII]. Recently a workable numerical procedure has been developed for the general case as well. The procedure augments the original set of points, P_j, by a set of unit vectors V_i where the c_i for maximizing form (1) associated with the points V_i are assumed "small." By choosing either V_i or $-V_i$, a feasible solution can be obtained by inspection rather than through the method of Section 2 of this paper. This cuts the computations in half. Moreover, the rank of the system is automatically m, i.e., $k = m$, so that by this approach all problems connected with degeneracy are solved.

and (6). They may be used to prove the following important propositions (actually, the proofs of Theorems A and B do not require the nondegeneracy assumption):

THEOREM A: *If one feasible solution exists, then there exists a feasible solution (called a basic feasible solution) with, at most, m points P_i with positive weights λ_i and $n - m$, or more, points P_i with $\lambda_i = 0$.*

THEOREM B: *If the values of z for the class of feasible solutions have a finite upper bound, then a maximum feasible solution exists which is a basic feasible solution.*

1. Construction of a Maximum Feasible Solution

Assume as given a feasible solution consisting of exactly m points, P_i, with nonzero weights; that is,

$$\lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_m P_m = P_0, \qquad \lambda_i > 0. \tag{7}$$

$$\lambda_1 c_1 + \lambda_2 c_2 + \cdots + \lambda_m c_m = z_0. \tag{8}$$

In establishing the conditions for and the construction of a maximum feasible solution, it will be necessary first to express all points, P_j, in terms of a *basis* consisting of m points which form the above feasible solution; that is,

$$x_{1j} P_1 + x_{2j} P_2 + \cdots + x_{mj} P_m = P_j \quad (j = 1, 2, \cdots, n). \tag{9}$$

We now define z_j by

$$x_{1j} c_1 + x_{2j} c_2 + \cdots + x_{mj} c_m = z_j \quad (j = 1, 2, \cdots, n). \tag{10}$$

THEOREM 1: *If, for any fixed j, the condition*

$$c_j > z_j \tag{11}$$

holds, then a set of feasible solutions can be constructed such that

$$z > z_0 \tag{12}$$

for any member of the set, where the upper bound of z is either finite or infinite.

CASE I: *If finite, a feasible solution consisting of exactly m points with positive weights can be constructed.*

CASE II: *If infinite, a feasible solution consisting of exactly $m + 1$ points with positive weights can be constructed such that the upper bound of z $= +\infty$.*

PROOF: Multiplying (9) by θ and subtracting from (7), and similarly multiplying (10) by θ and subtracting from (8), we get

$$(13)\quad (\lambda_1 - \theta x_{1j})P_1 + (\lambda_2 - \theta x_{2j})P_2 + \cdots + (\lambda_m - \theta x_{mj})P_m + \theta P_j = P_0,$$

$$(14)\quad (\lambda_1 - \theta x_{1j})c_1 + (\lambda_2 - \theta x_{2j})c_2 + \cdots + (\lambda_m - \theta x_{mj})c_m + \theta c_j = z_0 + \theta(c_j - z_j),$$

where the term θc_j has been added to both sides of (14).

Since $\lambda_i > 0$ for all i in (13), it is clear that there is, for $\theta \geqq 0$, either a finite range of values $\theta_0 > \theta \geqq 0$ or an infinite range of values such that the coefficients of P_i remain positive. It is clear from (14) that the z of this set of feasible solutions is a strictly monotonically increasing function of θ,

$$(15)\qquad z = z_0 + \theta(c_j - z_j) > z_0, \qquad \theta > 0,$$

since $c_j > z_j$ by hypothesis (11), thus establishing (12).

CASE I: If $x_{ij} > 0$ for at least one $i = 1, 2, \cdots, m$ in (13) or (9), the largest value of θ for which all coefficients in (13) remain nonnegative is given by

$$(16)\qquad \theta_0 = \min(\lambda_i/x_{ij}), \qquad x_{ij} > 0.$$

If $i = i_0$ yields θ_0 in (15), it is clear that the coefficient corresponding to i_0 in (13) and (14) will vanish, hence *a feasible solution, given by $\theta = \theta_0$, has been constructed with exactly m positive weights; moreover, $z > z_0$.* It will be noted that this new set of m points consists of the new point, P_j, and $(m - 1)$ of the m points previously used. This, then, is a desired solution for Case I of Theorem 1.

The new set of m points may be used as a *new basis*, and again, as in (9) and (10), all points may be expressed in terms of the new basis and the values of c_j compared with newly computed z_j's. If any $c_j > z_j$, the value of z can be increased. If at least one $x_{ij} > 0$, another new basis can be formed. We shall assume that the process is iterated until it is not possible to form a new basis. This must occur in a *finite* number of steps because, of course, there are at most $\binom{n}{m}$ bases and none of these bases can recur, for in that case their z-values would also recur, whereas the process gives strictly increasing values of z. Thus it is clear that the iteration must eventually terminate, either because at some stage

$$x_{ij} \leqq 0 \quad \text{for all} \quad i = 1, 2, \cdots, m \tag{17}$$

and some fixed j, or because

$$c_j \leqq z_j \quad \text{for all} \quad j = 1, 2, \cdots, n. \tag{18}$$

CASE II: If (17) holds (i.e., for all i, $x_{ij} \leqq 0$), then it is clear that θ has no finite upper bound and that a class of feasible solutions has been constructed consisting of $m + 1$ points with nonzero weights such that the upper bound of $z = +\infty$.

In all problems in which there is a finite upper bound to z, the iterative process must necessarily lead to condition (18). We shall prove, however, that the feasible solution associated with the final basis, which has the property $c_j \leqq z_j$ for all $j = 1, 2, \cdots, m$, is also a maximum feasible solution (Theorem 2). Hence, *in all problems in which there is no finite upper bound to z, the iterative process must necessarily lead to condition* (17); moreover, by rewriting (9) as

$$P_j + (-x_{1j})P_1 + (-x_{2j})P_2 + \cdots + (-x_{mj})P_m = 0, \qquad x_{ij} \leqq 0, \tag{19}$$

for the fixed j of (17), we have shown that *a nonnegative linear combination of* $(m + 1)$ *points vanishes if the upper bound of z is* $+\infty$. In many practical problems physical considerations will dictate the impossibility of (19).

As a practical computing matter the iterative procedure of shifting from one basis to the next is not as laborious as would first appear because the basis, except for the deletion of one point and the insertion of a new point, is the same as before. In fact, a shift of a basis involves less than mn multiplications and an equal number of additions. It has been observed *empirically* that the number of shifts of basis can be greatly reduced not by arbitrarily selecting any point, P_j, satisfying $c_j > z_j$, but by selecting the one which gives the greatest immediate increase in z; from (15) the criterion for choice of j is such that

$$\theta_0(c_j - z_j) = \max_j, \tag{20}$$

where θ_0 is given by (16) and is a function of j. A criterion that involves considerably less computation and apparently yields just as satisfactory results is to choose j such that

$$(c_j - z_j) = \max_j. \tag{21}$$

By the use of either (20) or (21) approximately m changes in basis are encountered in practice, so that about m^2n multiplications are involved in getting a maximum feasible solution from a feasible solution. There exist further refinements of computations by which $2m^2 + n$ computations are required per shift in basis if criterion (21) is used, or roughly $2m^3 + mn$ in all. However, to obtain a feasible solution will also require about $2m^3 + mn$ multiplications if one such solution is not readily available, and the selection of an original basis will require m^3 more—hence the method involves about $5m^3 + 2mn$ multiplications.[3]

THEOREM 2: *If, for all $j = 1, 2, \cdots, n$, the condition $c_j \leqq z_j$ holds, then* (7) *and* (8) *constitute a maximum feasible solution.*

PROOF: Let

$$\mu_1 P_1 + \mu_2 P_2 + \cdots + \mu_n P_n = P_0, \qquad \mu_j \geqq 0, \tag{22}$$

$$\mu_1 c_1 + \mu_2 c_2 + \cdots + \mu_n c_n = z^*, \tag{23}$$

constitute any other feasible solution. We shall show that $z_0 \geqq z^*$.

By hypothesis, $c_j \leqq z_j$, so that replacing c_j by z_j in (23) yields

$$\mu_1 z_1 + \mu_2 z_2 + \cdots + \mu_n z_n \geqq z^*. \tag{24}$$

Substituting the value of P_j given by (9) into (22) and the value of z_j given by (10) into (24), we obtain

$$\left(\sum_{j=1}^{n} \mu_j x_{1j}\right) P_1 + \left(\sum_{j=1}^{n} \mu_j x_{2j}\right) P_2 + \cdots + \left(\sum_{j=1}^{n} \mu_j x_{mj}\right) P_m = P_0, \tag{25}$$

$$\left(\sum_{j=1}^{n} \mu_j x_{1j}\right) c_1 + \left(\sum_{j=1}^{n} \mu_j x_{2j}\right) c_2 + \cdots + \left(\sum_{j=1}^{n} \mu_j x_{mj}\right) c_m \geqq z^*. \tag{26}$$

According to our assumption of nondegeneracy, the corresponding coefficients of P_i in (7) and (25) must be equal; hence (26) becomes

$$\lambda_1 c_1 + \lambda_2 c_2 + \cdots + \lambda_m c_m \geqq z^*; \tag{27}$$

or, by (8),

$$z_0 \geqq z^*. \tag{28}$$

In order that another maximum feasible solution exist it is necessary that $c_j = z_j$ for some P_j (not in the final basis). It will be noted, however, that in this case the extended matrix

$$\begin{bmatrix} P_1 & P_2 & \cdots & P_n \\ c_1 & c_2 & \cdots & c_n \end{bmatrix} \tag{29}$$

[3] See footnote 2 on page 340.

[see (4) above] has at least one set of $m + 1$ columns which are linearly dependent. *Thus a sufficient condition that the maximum feasible solution constructed from the given feasible solution be unique is that every set of* $(m + 1)$ *points, defined by columns in* (29), *be linearly independent.*

2. Construction of a Feasible Solution[4]

We begin by selecting an arbitrary basis of $(m - 1)$ points, P_j, and P_0. Denote this set by $(P_0; P_1, \cdots, P_{m-1})$. Any P_j can be expressed in terms of this basis by

$$(30) \qquad y_{0j}P_0 + y_{1j}P_1 + \cdots + y_{(m-1)j}P_{m-1} = P_j \quad (j = 1, 2, \cdots, m).$$

Theorem 3: *A sufficient condition that there exist no feasible solution is that* $y_{0j} \leqq 0$ *for all* j.

Proof: Assume on the contrary that there exists a feasible solution,

$$(31) \qquad \lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_n P_n = P_0, \qquad \lambda_j \geqq 0.$$

Substitute the expressions for P_j given by (30) into (31):

$$(32) \quad P_0\left(\sum_1^n \lambda_j y_{0j} - 1\right) + P_1\left(\sum_1^n \lambda_j y_{ij}\right) + \cdots + P_{m-1}\left(\sum_1^n \lambda_j y_{(m-1)j}\right) = 0.$$

In view of the assumed independence of $(P_0; P_1, \cdots, P_{m-1})$ it is clear that each coefficient in (32) must vanish; in particular,

$$(33) \qquad \sum_1^n \lambda_j y_{0j} - 1 = 0.$$

This is impossible if simultaneously $\lambda_j \geqq 0$ and $y_{0j} \leqq 0$ for all j.

To construct a feasible solution we first define a fixed reference point, G, given by

$$(34) \qquad G = w_1 P_1 + w_2 P_2 + \cdots + w_{m-1} P_{m-1} - \rho_0 P_0,$$

where $w_i > 0$ $(i = 1, \cdots, m - 1)$ and $\rho_0 > 0$ are arbitrarily chosen. For convenience we rewrite (34) in the form

$$(35) \qquad G + \rho_0 P_0 = w_1 P_1 + w_2 P_2 + \cdots + w_{m-1} P_{m-1}.$$

In the development that follows, ρ_0 will play a role analogous to z_0.

[4] See footnote 2 on page 340.

By Theorem 3, if there exists a feasible solution, there exists at least one j (which we shall consider fixed) such that

$$(36) \qquad y_{0j} > 0.$$

Multiplying (30) by θ and subtracting from (35), we obtain

$$(37) \quad G + (\rho_0 + \theta y_{0j})P_0 = \theta P_j + (w_1 - \theta y_{1j})P_1 + \cdots + (w_{m-1} - \theta y_{(m-1)j})P_{m-1}.$$

For a range of $\theta_0 > \theta > 0$ we can construct, in a manner analogous to (13) and (14), a set of points of the form $G + \rho P_0$, each given by a positive linear combination of points P_j. Since ρ will play a role analogous to z, we are interested in the highest value of ρ for which this is possible. It will be noted that

$$(38) \qquad \rho = \rho_0 + \theta y_{0j} > \rho_0$$

since $y_{0j} > 0$ has been assumed.

If, in the representation of P_j in (30), all $y_{ij} \leqq 0$ $(i = 1, \cdots, m - 1)$, the coefficients of P_j will be positive and $\rho \to +\infty$ as $\theta \to +\infty$. At the same time it will be seen, by solving (30) for P_0,

$$(39) \quad P_0 = (1/y_{0j})P_j + (-y_{1j}/y_{0j})P_1 + \cdots + (-y_{(m-1)j}/y_{0j})P_{m-1},$$

that a feasible solution has been obtained (i.e., P_0 has been expressed as a positive linear combination of $P_1, P_2, \cdots, P_{m-1}$ and P_j). If at least one $y_{ij} > 0$ $(i = 1, \cdots, m - 1)$, the largest value of θ is given by

$$(40) \qquad \theta_0 = \min_i (w_i/y_{ij}), \qquad y_{ij} > 0.$$

Setting $\theta = \theta_0$, the coefficient of at least one point, P_i, will vanish and a new point,

$$G + \rho_1 P_0,$$

will be formed from (34) which is expressed as a positive linear combination of just $m - 1$ points, P_i, where

$$(41) \qquad \rho_1 = \rho_0 + \theta_0 y_{0j} > \rho_0.$$

Expressing all points P_j in terms of the new basis, the process may be repeated, each time obtaining a higher value of ρ (or an infinite value, i.e., a feasible solution). The process must terminate in a finite number of steps. For, otherwise, since there is only a finite number of bases, the same combination of $(m - 1)$ points P_i would appear a second time; that is,

$$(42) \qquad G + \rho' P_0 = w_1' P_1 + w_2' P_2 + \cdots + w_{m-1}' P_{m-1},$$

$$(43) \qquad G + \rho'' P_0 = w_1'' P_1 + w_2'' P_2 + \cdots + w_{m-1}'' P_{m-1},$$

where $\rho'' > \rho'$. Subtracting (42) from (43), we obtain a nonvanishing expression giving P_0 in terms of $(m - 1)$ points P_i, contradicting the nondegeneracy assumption.

There are, however, only two conditions which will terminate the process; i.e., after a finite number of iterations either

$$(44) \qquad y_{0j} \leqq 0 \quad \text{for all} \quad j = 1, \cdots, n,$$

in which case, by Theorem 3, no feasible solution exists; or, for some fixed j,

$$(45) \qquad y_{ij} \leqq 0 \quad \text{for all} \quad i = 1, \cdots, m,$$

in which case, by solving (30) for P_0, as was done in (40), we obtain the desired feasible solution.

The term "simplex" technique arose in a geometric version of this development which assumes that one of the m equations (3) is of the form

$$(46) \qquad \lambda_1 + \lambda_2 + \cdots + \lambda_n = 1.$$

A point, P_j, is defined by the remaining coordinates in a column including c_j from (1) as an additional "z"-coordinate. We may interpret (1) and (3) as defining the center of gravity of a system of points P_j with weights λ_j. The problem consists, then, in finding weights λ_j so that the center of gravity lies on a line L defined by $m - 1$ of the relationships $x_1 = b_1$, $x_2 = b_2$, $\cdots$, $x_m = b_m$, such that the z-coordinate is maximum. A basis, $P_1, P_2, \cdots, P_m$, may be considered one of the faces of a simplex formed by $P_1, P_2, \cdots, P_m$ and P_j. The z-coordinate of P_j is c_j; the z-coordinate of the projection parallel to the z-axis of the point P_j on the plane of the face formed by the basis is z_j. Because $c_j > z_j$ by (11), all points in the simplex lie "above" the plane of this face. The line L cuts the base in an interior point whose z-value is z_0, hence it must intersect another face of the simplex in a "higher" point (i.e., a point whose z-value is greater than z_0).