Imperial College London

Course: M5MF2
Setter: Dr Antoine Jacquier

MSc EXAMINATIONS IN MATHEMATICS AND FINANCE
DEPARTMENT OF MATHEMATICS

April 2016

M5MF2

Numerical Methods in Finance

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Date: April 2016 Time:

Answer all questions.

The total number of points is 100, and the precise grading is indicated in the text.

The rigour and clarity of your answers will be taken into account in the final grade.

Each problem is independent of the others.

1 [10 points] Warm-up: Preliminary questions

(i) [5 points] Consider the function $f(x) := \mathrm{e}^{-x}$ for $x \geq 0$ and null elsewhere. Is f the density of some random variable? If so, compute the characteristic function ϕ of the corresponding random variable. Conversely, given ϕ can we recover f?

(ii) Let X be a random variable taking values $\{x_1, x_2\}$ with respective probabilities p and $1 - p$.

(a) [1 point] Compute its characteristic function.

(b) [4 points] Consider now the following characteristic function:

$$\Phi(\xi) = \frac{1}{8}\left(1 + 7\mathrm{e}^{\mathrm{i}\xi}\right).$$

By computing $|\Phi(\xi)| := \sqrt{\Phi(\xi)\overline{\Phi}(\xi)}$ (the overline representing the complex conjugate), show that the absolute value of a characteristic function is not necessarily a characteristic function itself.

2 [30 points] Pricing with Fourier transforms

We consider here a Power Call option written on an underlying stock price $(S_t)_{t\geq 0}$, with maturity $T > 0$ and log-strike $k \in \mathbb{R}$, which has the following payoff: $(S_T^p - \mathrm{e}^k)_+$, for some $p > 0$. We shall assume that the random variable $X_T := \log(S_T)$ admits a density f, with support on $\mathbb{R}$, and with characteristic function $\Phi(\xi) := \mathbb{E}(\mathrm{e}^{\mathtt{i}\xi X_T})$ on $\mathbb{R}$. We assume (obviously) that the p-th moment of S_T is finite, that $S_0 = 1$ and that interest rates are null.

(i) [2 points] Consider the Fourier transform $\widehat{C}_p : \mathbb{R} \to \mathbb{C}$ of the Call price:

$$\widehat{C}_p(\xi) := \int_{\mathbb{R}} \mathrm{e}^{\mathtt{i}\xi k} C_p(k)\mathrm{d}k.$$

What is the problem with this definition?

(ii) [10 points] Assuming no-arbitrage, show that the price C_p of the Power Call option today is equal to

$$C_p(k) = \frac{\mathrm{e}^{-\alpha k}}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{-\mathtt{i}\xi k} \Psi_p(\xi)\mathrm{d}\xi,$$

where α is a dampening factor, the range of which needs to be made explicit, and where the function Ψ_p (which depends on α) should be written explicitly in terms of the characteristic function of X_T. *Hint: You may want to define the dampened Call price* $c_p(k) := \mathrm{e}^{\alpha k} C_p(k)$.

(iii) Unfortunately, this method notoriously loses accuracy when the maturity of the option becomes small. In order to remedy this, we propose another methodology, and consider the simple case of a standard Call option, with $p = 1$. Introduce the function $h : \mathbb{R} \to \mathbb{R}$, which corresponds to a Put price if $k < \log(S_0)$ and to a Call price if $k > \log(S_0)$. Its characteristic function reads

$$\widehat{h}(\xi) := \int_{\mathbb{R}} \mathrm{e}^{\mathtt{i}\xi k} h(k)\mathrm{d}k, \qquad \text{for all } \xi \in \mathbb{R}.$$

(a) [2 points] Is the characteristic function well defined here?

(b) [6 points] Show that, for any $\xi \in \mathbb{R}$, the following identity holds:

$$\widehat{h}(\xi) = \frac{1}{1 + \mathtt{i}\xi} - \frac{1}{\mathtt{i}\xi} - \frac{\Phi(\xi - \mathtt{i})}{\xi^2 - \mathtt{i}\xi}.$$

(iv) Now, for $k = 0$, when the maturity T is small, the function h_p becomes close to a Dirac function, and its Fourier transform becomes very oscillatory. We therefore consider its transform $g(k) \equiv \sinh(\alpha k)h(k)$.

(a) [2 points] What is the behaviour of the function g at $k = 0$?

(b) [6 points] Prove that the following equality holds for all real number k:

$$h(k) = \frac{1}{2\pi}\frac{1}{\sinh(\alpha k)}\int_{\mathbb{R}} \mathrm{e}^{-\mathtt{i}\xi k}\widehat{g}(\xi)\mathrm{d}\xi,$$

where

$$\widehat{g}(\xi) := \frac{1}{2}\left\{\widehat{h}(\xi - \mathtt{i}\alpha) - \widehat{h}(\xi + \mathtt{i}\alpha)\right\}, \qquad \text{for all } \xi \in \mathbb{R}.$$

(c) [2 points] How does the parameter α influence the integration?

3 [40 points] Finite difference methods for periodic PDEs

We consider here the following periodic heat equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \sigma \dfrac{\partial^2 u}{\partial x^2}, & \text{for } (x,t) \in (0,1) \times (0,\infty), \\ u(0,x) = f(x), & \text{for } x \in (0,1), \\ u(t,x+1) = u(t,x), & \text{for } (x,t) \in \mathbb{R} \times (0,\infty), \end{cases} \tag{1}$$

where f is a smooth function (i.e. of class $\mathcal{C}^\infty(\mathbb{R})$) and σ a strictly positive constant. Let N be a strictly positive integer and define the grid $\{(n\delta_t, j\delta_x)\}_{n\geq 0, j\in\mathbb{Z}}$, with $\delta_x := 1/N$ and $\delta_t > 0$ fixed. We further denote u_j^n the discrete approximation of the exact solution u at the point $(n\delta_t, j\delta_x)$. Let now α, β and γ be three real constants, independent of δ_x and δ_t, and consider the following finite difference scheme:

$$\frac{\alpha u_j^{n+1} + \beta u_j^n + \gamma u_j^{n-1}}{\delta_t} - \sigma \frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{\delta_x^2} = 0, \tag{2}$$

with appropriate boundary conditions. Note that this is a 3-level scheme, involving three time steps, and not two.

(i) [5 points] Using Taylor series, show that the scheme is consistent if and only if $\alpha = 1 + \gamma$ and $\beta = -1 - 2\gamma$. When $\gamma = -1$, is the scheme explicit or implicit? What do we know about the stability of the scheme in this case?

(ii) [3 points] Assume that $\gamma \in (-1/2, 0]$. Using geometric arguments based on convexity, formulate the Courant-Friedrichs-Lewy condition for the stability of the scheme involving γ, σ, δ_t and δ_x.

(iii) [8 points] We shall from now on assume that the scheme is consistent. Consider $\gamma \in (-\infty, -1) \cup (-1, -1/2)$, and for any $n \geq 0$, define the discrete Fourier transform

$$\widehat{u}^n(\xi) := \sum_{j\in\mathbb{Z}} \delta_x \mathrm{e}^{-ij\delta_x\xi} u_j^n, \qquad \text{for any } \xi \in \left[-\frac{\pi}{\delta_x}, \frac{\pi}{\delta_x}\right].$$

The inverse Fourier transform reads for any $j \in \mathbb{Z}$: $u_j^n = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\pi/\delta_x}^{\pi/\delta_x} \mathrm{e}^{\mathtt{i}j\delta_x\xi}\, \widehat{u}^n(\xi)\mathrm{d}\xi$. For any $\xi \in [-\pi/\delta_x, \pi/\delta_x]$, define the vector $\widehat{U}^n(\xi) \in \mathbb{R}^2$ by $\widehat{U}^n(\xi) := (\widehat{u}^{n+1}(\xi), \widehat{u}^n(\xi))'$ (where $'$ denotes the transpose of the vector), and prove for all $n \geq 0$, there exists a two-by-two matrix $A(\xi)$ such that $\widehat{U}^n(\xi) = A(\xi)\widehat{U}^{n-1}(\xi)$. Using the L^2-matrix norm, prove that the scheme is not stable.

(iv) [6 points] Using a similar analysis as in the previous question, investigate the stability of the scheme when $\gamma \geq 0$.

(v) We are now interested in a two-level finite difference scheme for the heat equation (1) above.

(a) $[5$ points$]$ Write down an implicit scheme for the heat equation (in matrix form).

(b) $[5$ points$]$ For real numbers a, b and c, such that $bc > 0$, consider the tridiagonal matrix

$$\mathrm{T} := \begin{pmatrix} a & c & 0 & 0 \\ b & a & \ddots & 0 \\ 0 & \ddots & \ddots & c \\ 0 & 0 & b & a \end{pmatrix} \in \mathcal{M}_n(\mathbb{R}),$$

and recall that T has n eigenvalues given by

$$\lambda_k = a + 2\sqrt{bc}\cos\left(\frac{\pi k}{n+1}\right), \qquad \text{for } k = 1, \ldots, n.$$

Determine the eigenvalues of the transition matrix in the implicit scheme and using the 2-norm, conclude on the convergence on the scheme.

4 [20 points] Existence of a solution to the heat equation

We are interested here in proving the existence of a solution to the heat equation (on the real line) given some initial data. Consider the heat equation

$$\frac{\partial \phi}{\partial t} = \frac{1}{2}\frac{\partial^2 \phi}{\partial x^2}, \qquad \text{for } x \in \mathbb{R}, t \geq 0, \tag{3}$$

with boundary condition $\phi(x, 0) = f(x)$ for any $x \in \mathbb{R}$. Let now H_t be the function defined by

$$H_t(x) := \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right), \qquad \text{for all } x \in \mathbb{R}, t \geq 0,$$

and define the function $u : \mathbb{R} \times [0, \infty)$ as the convolution $u(x, t) := (f * H_t)(x)$

(i) [5 points] Recall the definition of the Schwartz space $\mathcal{S}(\mathbb{R})$ and of an $L^p(\mathbb{R})$ space ($p \geq 1$); is one included in the other? For $p = 2$, Give two examples of functions (supported on the whole real line) each belonging to each spaces, and two examples not belonging to these two spaces.

(ii) [5 points] Prove that if $f \in \mathcal{S}(\mathbb{R})$, then u is of class $\mathcal{C}^2(\mathbb{R})$ when $t > 0$. What happens at $t = 0$? Prove that u solves the heat equation (3).

(iii) [5 points] Using Plancherel's formula, prove that $\int_{\mathbb{R}} |u(x, t) - f(x)|^2 \mathrm{d}x$ tends to zero as t tends to zero.

(iv) [5 points] Finally, assuming that f is a Schwartz function, prove that for any $t > 0$, the function $x \mapsto u(x, t)$ also belongs to $\mathcal{S}(\mathbb{R})$. Hint: Take $z > 0$ and split the domain of integration defining u into $[-z, z]$ and $\mathbb{R} \setminus [-z, z]$.