MSC COURSE IN MATHEMATICS AND FINANCE

Numerical Methods in Finance
Final Exam, May 2015

Answer all questions. Each problem is independent of the others. The total number of points is 100, and the number of points for each question is indicated. The rigour and clarity of your answers will be taken into account in the final grade.

1. Warm-up: Preliminary questions [15 points]

(i) [3 points] For a real number a and two smooth functions f and g from $\mathbb{R}$ to $\mathbb{R}$, write the exact mathematical definition of the following statements:
  (a) $f(x) = \mathcal{O}(g(x))$, $\quad (x \to a)$;
  (b) $f(x) = o(g(x))$, $\quad (x \to a)$;
  (c) $f(x) = \mathcal{O}(g(x))$, $\quad (x \to +\infty)$;

(ii) [3 points] For a random variable X with support equal to the whole real line, define its characteristic function ϕ_X as well as its domain. If X admits a density f, write ϕ_X in terms of f.

(iii) [4 points] Consider the function $f(x) := \mathrm{e}^{-x}$ for $x \geq 0$ and null elsewhere. Is f the density of some random variable? If so, compute the characteristic function ϕ of the corresponding random variable. Conversely, given ϕ can we recover f?

(iv) [5 points] Let X be a random variable taking values on the whole real line, and ϕ its characteristic function. Prove that, if $\mathbb{E}(|X|^n)$ is finite, then the n-th derivative $\phi^{(n)}$ exists and $\mathbb{E}(X^n) = (-\mathtt{i})^n \phi^{(n)}(0)$.

2. Fourier transform and interpolation [30 points]

We consider here a 'direct' method to solve the heat equation. By direct, we mean, first an integral solution, then a numerical implementation of this integral.

(i) [8 points] Fix $\sigma > 0$ and consider the heat equation $\partial_\tau \phi_\tau = \frac{1}{2}\sigma^2 \partial_{xx}^2 \phi_\tau$ for $x \in \mathbb{R}, \tau > 0$, with boundary condition $\phi_0(x) = f(x)$, and that the function f is smooth and has linear growth, i.e. there exists $\gamma > 0$ such that $|f(x)| \leq \gamma(1 + |x|)$ for any $x \in \mathbb{R}$. For $\tau > 0$, define the Fourier transform $\widehat{\phi}_\tau$ of the function ϕ_τ by

$$\widehat{\phi}_\tau(z) := \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}zx}\phi_\tau(x)\mathrm{d}x, \qquad \text{for any } z \in \mathbb{R}.$$

Using integrations by parts, compute $\widehat{\partial_{xx}\phi_\tau}$, i.e. the Fourier transform of the second derivative in space of the function ϕ_τ. Deduce that

$$\widehat{\phi}_\tau(z) = \widehat{f}(z)\exp\left(-\frac{\sigma^2 z^2 \tau}{2}\right), \qquad \text{for any } z \in \mathbb{R},$$

and hence that

$$\phi_\tau(x) = \frac{1}{\sigma\sqrt{2\pi\tau}}\int_{\mathbb{R}} f(\xi)\exp\left(-\frac{(x-\xi)^2}{2\sigma^2\tau}\right)\mathrm{d}\xi, \qquad \text{for any } x \in \mathbb{R}. \tag{2.1}$$

You may assume without proof that the functions ϕ_τ and $\partial_x\phi_\tau$ converge to zero at infinity.

(ii) [7 points] Consider now a scaled form of the integral in (2.1):

$$I := \int_{\mathbb{R}} f(\xi)\exp\left(-\frac{\xi^2}{2}\right)\mathrm{d}\xi. \tag{2.2}$$

Consider the approximation

$$I_z := \int_{-z}^{z} f(\xi)\exp\left(-\frac{\xi^2}{2}\right)\mathrm{d}\xi, \tag{2.3}$$

for some $z > 0$. Show that the error $|I - I_z|$ tends to zero as z tends to infinity.

(iii) We now wish to determine an interpolation scheme for the integral (2.3). Consider $n+1$ ($n \geq 1$) nodes $-z \leq x_0 \leq \ldots \leq x_n \leq z$, and construct a polynomial $P(x) \equiv \sum_{k=0}^n \alpha_k x^k$, for some coefficients $(\alpha_0, \ldots, \alpha_n)$ such that $P(x_k) = \psi(x_j)$, where we define the function ψ by $\psi(x) \equiv f(x)\exp\left(-\frac{1}{2}x^2\right)$.

(a) [2 points] Write down a matrix equation that the coefficients $(\alpha_0, \ldots, \alpha_n)$ have to solve.

(b) [7 points] Consider the matrix

$$\mathrm{M} := \begin{pmatrix} 1 & x_0 & \cdots & x_0^n \\ 1 & x_0 & \cdots & x_0^n \\ \vdots & \vdots & & \vdots \\ 1 & x_n & \cdots & x_n^n \end{pmatrix}.$$

Prove by induction that the determinant of this (Vandermonde) matrix reads

$$\det(\mathrm{M}) = \prod_{i>j}(x_i - x_j).$$

(c) [1 point] What can you deduce about the existence/uniqueness of the interpolation problem?

(iv) [5 points] We now consider Lagrange polynomials. Write down an approximation of $\Phi(\xi)$ using interpolating Lagrange polynomials at the nodes $x_0, \ldots, x_n$. Deduce an approximation formula for I in (2.2).

3. Finite difference methods for a periodic PDE [40 points]

We consider here the following periodic heat equation:

$$
\left\{
\begin{array}{ll}
\dfrac{\partial u}{\partial t} = \sigma \dfrac{\partial^2 u}{\partial x^2}, & \text{for } (x,t) \in (0,1) \times \mathbb{R}_+^*, \\
u(0,x) = f(x), & \text{for } x \in (0,1), \\
u(t,x+1) = u(t,x), & \text{for } (x,t) \in \mathbb{R} \times \mathbb{R}_+^*,
\end{array}
\right.
\tag{3.1}
$$

where f is a smooth function (i.e. of class $\mathcal{C}^\infty(\mathbb{R})$) and σ a strictly positive constant. Let N be a strictly positive integer and define the grid $\{(n\delta_t, j\delta_x)\}_{n\geq 0, j\in\mathbb{Z}}$, with $\delta_x := 1/N$ and $\delta_t > 0$ fixed. We further denote u_j^n the discrete approximation of the exact solution u at the point $(n\delta_t, j\delta_x)$. Let now α, β and γ be three real constants, independent of δ_x and δ_t, and consider the following finite difference scheme:

$$
\frac{\alpha u_j^{n+1} + \beta u_j^n + \gamma u_j^{n-1}}{\delta_t} - \sigma \frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{\delta_x^2} = 0,
\tag{3.2}
$$

with appropriate boundary conditions. Note that this is a 3-level scheme, involving three time steps, and not two.

(i) [5 points] Using Taylor series, show that the scheme is consistent if and only if $\alpha = 1+\gamma$ and $\beta = -1-2\gamma$. When $\gamma = -1$, is the scheme explicit or implicit? What do we know about the stability of the scheme in this case?

(ii) [3 points] Assume that $\gamma \in (-1/2, 0]$. Using geometric arguments based on convexity, formulate the Courant-Friedrichs-Lewy condition for the stability of the scheme involving γ, σ, δ_t and δ_x.

(iii) [8 points] Consider now $\gamma \in (-\infty,-1) \cup (-1,-1/2)$, and for any $n \geq 0$, define the discrete Fourier transform

$$
\widehat{u}^n(\xi) := \sum_{j\in\mathbb{Z}} \delta_x \mathrm{e}^{-ij\delta_x\xi} u_j^n, \qquad \text{for any } \xi \in \left[-\frac{\pi}{\delta_x}, \frac{\pi}{\delta_x}\right].
$$

The inverse Fourier transform reads for any $j \in \mathbb{Z}$:

$$
u_j^n = \frac{1}{\sqrt{2\pi}} \int_{-\pi/\delta_x}^{\pi/\delta_x} \mathrm{e}^{\mathtt{i}j\delta_x\xi}\, \widehat{u}^n(\xi) \mathrm{d}\xi.
$$

For any $\xi \in [-\pi/\delta_x, \pi/\delta_x]$, define the vector $\widehat{U}^n(\xi) \in \mathbb{R}^2$ by $\widehat{U}^n(\xi) := (\widehat{u}^{n+1}(\xi), \widehat{u}^n(\xi))'$ (where $'$ denotes the transpose of the vector), and prove for all $n \geq 0$, there exists a two-by-two matrix $A(\xi)$ such that $\widehat{U}^n(\xi) = A(\xi)\widehat{U}^{n-1}(\xi)$. Using the L^2-matrix norm, prove that the scheme is unconditionally unstable.

(iv) [6 points] Using a similar analysis as in the previous question, investigate the stability of the scheme when $\gamma \geq 0$.

We are now interested in a two-level finite difference scheme for the heat equation (3.1) above.

(v) [5 points] Write down an implicit scheme for the heat equation (in matrix form).

(vi) [8 points] For three real numbers a, b and c, consider the following matrix in $\mathcal{M}_n(\mathbb{R})$:

$$
\mathrm{T} := \begin{pmatrix} a & c & 0 & 0 \\ b & a & \ddots & 0 \\ 0 & \ddots & \ddots & c \\ 0 & 0 & b & a \end{pmatrix} \in \mathcal{M}_n(\mathbb{R}).
$$

Prove from first principles (not by checking that the expressions are valid) that T has n eigenvalues given by

$$
\lambda_k = a + 2\sqrt{bc}\cos\left(\frac{\pi k}{n+1}\right), \qquad \text{for } k = 1, \dots, n.
$$

(vi) [5 points] Determine the eigenvalues of the transition matrix in the implicit scheme and using the 2-norm, conclude on the convergence on the scheme.

4. Implied volatility [15 points]

We are now interested in the computation of the implied volatility, which is an essential task in the daily life of quantitative analysts. We consider a stock price process $(S_t)_{t\geq 0}$. We assume that we are given a fixed strike K and maturity $T > 0$ for which we observe a call option value $C^*(K,T)$ from the market.

(i) [2 points] Define what the implied volatility is.

(ii) [3 points] Let $C_{\text{BS}}(S_0, K, T, \sigma)$ denote the price of a European Call option with strike K and maturity T in the Black-Scholes model with no interest rate and volatility coefficient σ. State the Black-Scholes formula and prove that the map $\sigma \mapsto C_{\text{BS}}(S_0, K, T, \sigma)$ is strictly increasing for any $K > 0$, $T > 0$.

(iii) [3 points] Compute the limit of $C_{\text{BS}}(S_0, K, T, \sigma)$ as σ tends to zero and to infinity.

(iv) [4 points] Explain how to apply Newton's method to compute the implied volatility $\sigma(K,T)$, and prove the speed of convergence of the method.

(v) [3 points] What are the potential pitfalls of this root-finding algorithm? When can this happen in the present computation?