MSC COURSE IN MATHEMATICS AND FINANCE

Numerical Methods in Finance
Final Exam, June 2014

Answer all questions. Each problem is independent of the others. The total number of points is 80, and the number of points for each question is indicated. The rigour and clarity of your answers will be taken into account in the final grade.

1. [10 POINTS] WARM-UP: PRELIMINARY QUESTIONS

(i) [3 points] For a real number a and two smooth functions f and g from $\mathbb{R}$ to $\mathbb{R}$, write the exact mathematical definition of the following statements:

(a) $f(x) = \mathcal{O}(g(x)), \quad (x \to a)$;

(b) $f(x) = o(g(x)), \quad (x \to a)$;

(c) $f(x) = \mathcal{O}(g(x)), \quad (x \to +\infty)$;

(ii) [3 points] For a random variable X with support equal to the whole real line, define its characteristic function ϕ_X. If X admits a density f, write ϕ_X in terms of f. Is ϕ_X well defined?

(iii) [4 points] Consider the function $f(x) := \mathrm{e}^{-x}$ for $x \geq 0$ and null elsewhere. Is f the density of some random variable? If so, compute its characteristic function ϕ. Conversely, given ϕ can we recover f?

2. [20 points] Fourier transform and interpolation

We consider here a 'direct' method to solve the heat equation. By direct, we mean, first an integral solution, then a numerical implementation of this integral.

(i) [7 points] Fix $\sigma > 0$ and consider the heat equation $\partial_\tau \phi_\tau = \frac{1}{2}\sigma^2 \partial_{xx}^2 \phi_\tau$ for $x \in \mathbb{R}, \tau > 0$, with boundary condition $\phi_0(x) = f(x)$ for some continuous function f. For fixed $\tau > 0$, define the Fourier transform $\widehat{\phi}_\tau$ of the function ϕ_τ by

$$\widehat{\phi}_\tau(z) := \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{\mathtt{i}zx}\phi_\tau(x)\mathrm{d}x, \qquad \text{for any } z \in \mathbb{R}.$$

Using integrations by parts, compute $\widehat{\partial_{xx}\phi_\tau}$, i.e. the Fourier transform of the second derivative in space of the function ϕ_τ. Deduce that

$$\widehat{\phi}_\tau(z) = \widehat{f}(z)\exp\left(-\frac{\sigma^2 z^2 \tau}{2}\right), \qquad \text{for any } z \in \mathbb{R},$$

and hence that

$$\phi_\tau(x) = \frac{1}{\sigma\sqrt{2\pi\tau}}\int_{\mathbb{R}} f(\xi)\exp\left(-\frac{(x-\xi)^2}{2\sigma^2\tau}\right)\mathrm{d}\xi, \qquad \text{for any } x \in \mathbb{R}. \tag{2.1}$$

You may assume without proof that the functions ϕ_τ and $\partial_x\phi_\tau$ converge to zero at infinity.

(ii) [7 points] Consider now a scaled form of the integral in (2.1):

$$I := \int_{\mathbb{R}} \Phi(\xi)\mathrm{d}\xi, \tag{2.2}$$

where $\Phi(\xi) := f(\xi)\exp\left(-\xi^2/2\right)$. Assume that the function f has linear growth, i.e. there exists $\gamma > 0$ such that $|f(x)| \leq \gamma(1+|x|)$ for any $x \in \mathbb{R}$. Consider the approximation

$$I_z := \int_{-z}^{z} f(\xi)\exp\left(-\frac{\xi^2}{2}\right)\mathrm{d}\xi,$$

for some $z > 0$. Determine an upper bound for the error $|I - I_z|$.

(iii) [6 points] Using $n+1$ ($n \geq 1$) nodes $-z \leq x_0 \leq \ldots \leq x_n \leq z$, write down an approximation of $\Phi(\xi)$ using interpolating Lagrange polynomials at the nodes $x_0, \ldots, x_n$. Deduce an approximation formula for I in (2.2).

3. [30 POINTS] θ-SCHEMES AND STABILITY

We consider here a stock price process $(S_t)_{t\geq 0}$ that satisfies the following stochastic differential equation (Black-Scholes model) under the risk-neutral probability:

$$\frac{\mathrm{d}S_t}{S_t} = r\mathrm{d}t + \sigma \mathrm{d}W_t, \qquad \text{with } S_0 > 0.$$

The risk-free interest rate $r \geq 0$ and the volatility $\sigma > 0$ are assumed to be constant. Let $V_t(S)$ denote the value at time t of a European option on S. From the Feynman-Kac formula, we know that V_t satisfies the following partial differential equation:

$$\partial_t V_t(S) + rS\partial_S V_t(S) + \frac{\sigma^2}{2} S^2 \partial_{SS} V_t(S) - rV_t(S) = 0. \tag{3.1}$$

A double knock-out European barrier call option with lower barrier $L > 0$, upper barrier $U > 0$, strike $K > 0$ and maturity T has the following payoff at maturity:

$$V_T(S) = (S_T - K)_+ \mathbb{1}_{\left\{\inf_{0\leq t\leq T}\{S_t\}>L\right\}} \mathbb{1}_{\left\{\sup_{0\leq t\leq T}\{S_t\}<U\right\}}.$$

The goal of this problem is to determine the price of such an option and implement a finite-difference scheme to evaluate it.

(i) [2 points] Determine the type of Equation (3.1) and write the boundary conditions that the barrier option has to satisfy.

(ii) [7 points] Using appropriate changes of variables, reduce Equation (3.1) into the heat equation

$$\partial_\tau \Pi_\tau(x) = \alpha \partial_{xx} \Pi_\tau(x), \tag{3.2}$$

where $\alpha > 0$ is some constant to be determined and write explicitly the corresponding boundary conditions.

(iii) We now wish to write an explicit finite difference scheme for the PDE (3.1). Let n be the discretisation step in the time direction, so that $[0, T] = [0, \delta_T, 2\delta_T, \ldots, n\delta_T]$ with $\delta_T := T/n$; Fix some large value $\overline{S} > 0$, so that the closed interval $[0, \overline{S}]$ represents the truncation of the space domain $\mathbb{R}_+$, and let m be the space increment size: $[0, \overline{S}] = [0, \delta_S, 2\delta_S, \ldots, m\delta_S]$, with $\delta_S := \overline{S}/m$.

- (a) [10 points] Write down an explicit scheme for (3.1).
- (b) [3 points] Determine the accuracy of the scheme.
- (c) [5 points] Prove that a necessary condition ensuring the stability of the scheme is that the norm of the iteration matrix is strictly smaller than one. Would it be simpler to work directly with the reduced form (3.2)?

(iv) [3 points] Should you construct a finite-difference scheme for a standard European Put option (no barrier). What kind of boundary conditions could you specify at $S_t = 0$ and $S_t = \overline{S}$, for any $t \in (0, T)$?

4. [20 points] Existence of a solution to the heat equation

We are interested here in proving the existence of a solution to the heat equation (on the real line) given some initial data. Consider the heat equation

$$\partial_t \phi = \partial_{xx}^2 \phi, \qquad \text{for } x \in \mathbb{R}, t > 0, \tag{4.1}$$

with initial boundary condition $\phi(x, 0) = f(x)$ for any $x \in \mathbb{R}$. Let now H_t be the function defined by

$$H_t(x) := \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right), \qquad \text{for all } x \in \mathbb{R}, t > 0,$$

and H_0 is the Dirac function at 0. Define the function $u : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ as the convolution

$$u(x, t) := (f * H_t)(x) := \int_{\mathbb{R}} f(x - y) H_t(y) \mathrm{d}y, \qquad \text{for all } x \in \mathbb{R}, t \geq 0.$$

(i) [2 points] Recall the definitions of the Schwartz space $\mathcal{S}(\mathbb{R})$ and an $L^p(\mathbb{R})$ space ($p \geq 1$).

(ii) [6 points] Prove that if $f \in \mathcal{S}(\mathbb{R})$, then u is of class $\mathcal{C}^2(\mathbb{R})$ when $t > 0$. Prove that u solves the heat equation (4.1).

(iii) [6 points] Using Plancherel's formula, prove that $\int_{\mathbb{R}} |u(x, t) - f(x)|^2 \mathrm{d}x$ tends to zero as t tends to zero.

(iv) [6 points] Finally, assuming that f is a function in the Schwartz space, prove that for any $t > 0$, the function $x \mapsto u(x, t)$ also belongs to $\mathcal{S}(\mathbb{R})$. Hint: Take $z > 0$ and split the domain of integration defining u into $[-z, z]$ and $\mathbb{R} \setminus [-z, z]$.