Heat Equation - FD

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1 Finite-difference scheme for the one-dimensional heat equation

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In [3]:
```
from time import time
# For sparse matrices
from scipy.sparse import dia_matrix
from scipy.sparse.linalg.dsolve import spsolve
```

We consider here the heat equation on $[0, \overline{x}] \times [0, \infty)$:

$$\frac{\partial u}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2},$$

with boundary conditions $u(x, 0) = \sin(2\pi x)$ for all $x \in [0, \overline{x}]$ and $u(0, t) = u(\overline{x}, t) = 0$ for all $t \geq 0$. We apply a finite difference scheme, explicit in time and with central difference in space

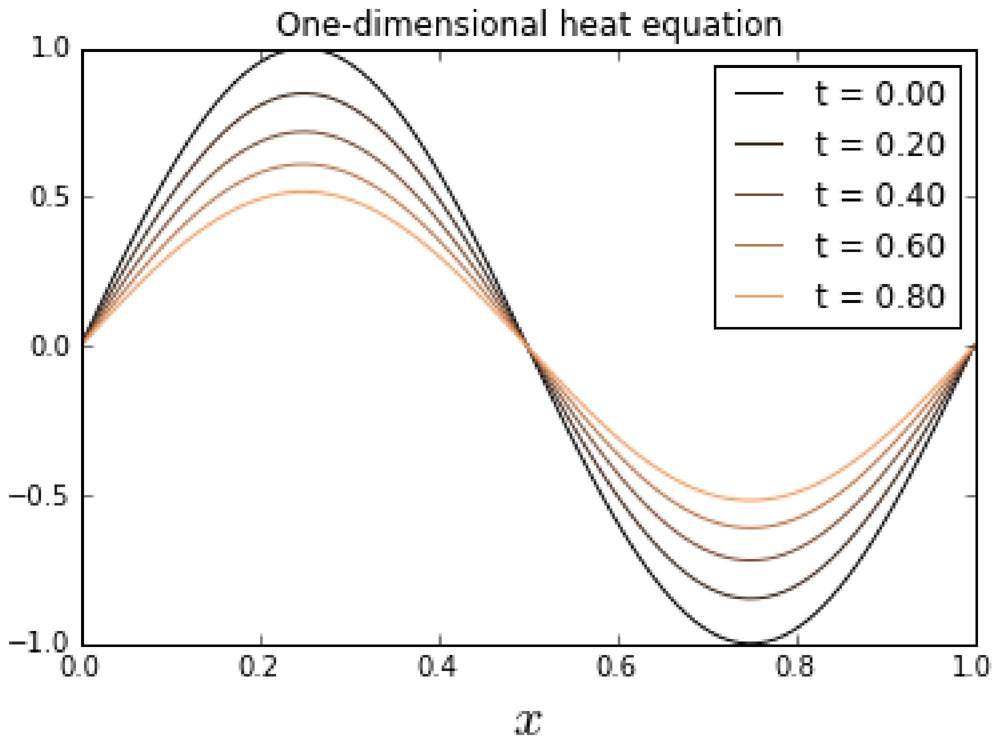
2 Explicit scheme in time

2.1 No vector notation

In [4]:
```
sigma, barX, T = 0.2, 1., 1.
nx = 100  # Number of grid points
nt = 5000  # Number of time steps
```

In [5]:
```
start_time = time()

dx = barX / (nx - 1)  # Grid step in space
dt = T / nt          # Grid step in time
print 'Ratio $\delta_T \sigma^2/ \delta_x^2$ = ', (sigma * sigma * dt / (dx * dx))
# Boundary conditions
x = linspace(0.0, barX, nx)
u = sin(2.0 * pi * x)

for n in range(0, nt):

   for j in range(1, nx - 1):
      u[j] += dt * (0.5 * sigma * sigma) * \
          (u[j - 1] - 2 * u[j] + u[j + 1]) / (dx ** 2)

   # Plot every pp time steps
   pp = 100
   if (n % pp == 0):
      if (n % (10 * pp) == 0):
         plotlabel = "t = %1.2f" % (n * dt)
         plot(x, u, label=plotlabel, color=get_cmap(
             'copper')(float(n) / nt))

xlabel(u'$x$', fontsize=20)
title(u'One-dimensional heat equation')
legend()
show()
temp = time() - start_time
print("--- Computation time: %s seconds ---" % temp)
```

```
Ratio $\delta_T \sigma^2/ \delta_x^2$ =  0.078408
```

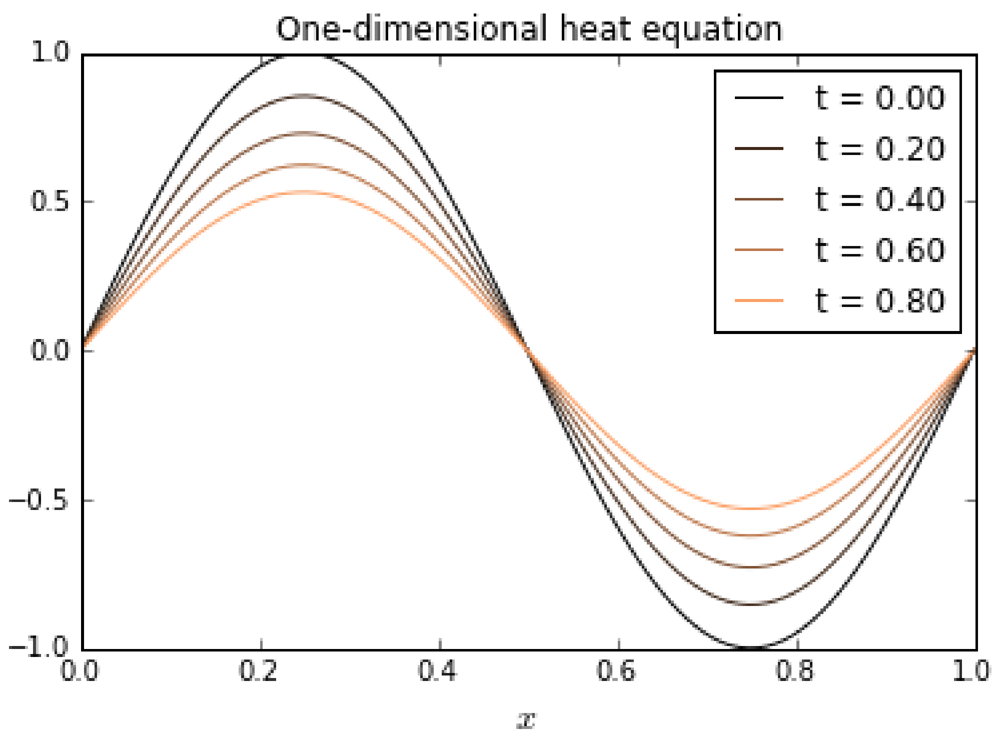



```
--- Computation time: 2.79689407349 seconds ---
```

2.2 Vector notations

```
In [6]: start_time = time()

        dx = barX / (nx - 1)  # Grid step in space
        dt = T / nt           # Grid step in time

        # Boundary conditions
        x = linspace(0.0, barX, nx)
        u = sin(2.0 * pi * x)
        rhs = zeros(nx)
        for n in range(0, nt):
           rhs[1:-1] = dt * (0.5 * sigma * sigma) * \
               (u[:-2] - 2.0 * u[1:-1] + u[2:]) / (dx * dx)
           u += rhs
           # Plot every pp time steps
           pp = 100
           if (n % pp == 0):
```

```
        if (n % (10 * pp) == 0):
            plotlabel = "t = %1.2f" % (n * dt)
            plot(x, u, label=plotlabel, color=get_cmap(
                'copper')(float(n) / nt))

xlabel(u'$x$', fontsize=14)
title(u'One-dimensional heat equation')
legend()
show()
temp = time() - start_time
print("--- Computation time: %s seconds ---" % temp)
```




```
--- Computation time: 0.532827854156 seconds ---
```

2.3 Matrix notations, using sparse matrices

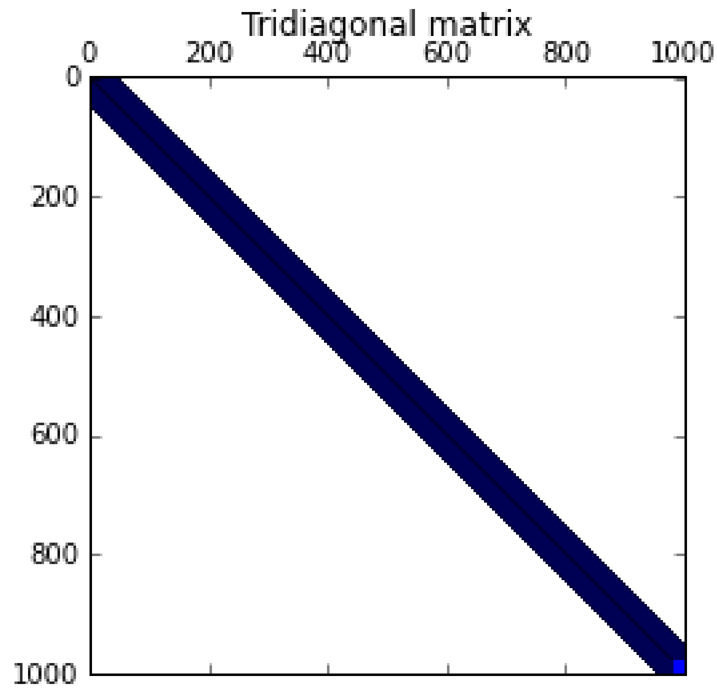
We are interested here in solving the matrix system in $\mathbb{R}^N$:

$$\mathrm{Diag}(-2, 1, 1)\mathbf{u} = (\mathrm{d}x)^2(1, \ldots, 1)'.$$

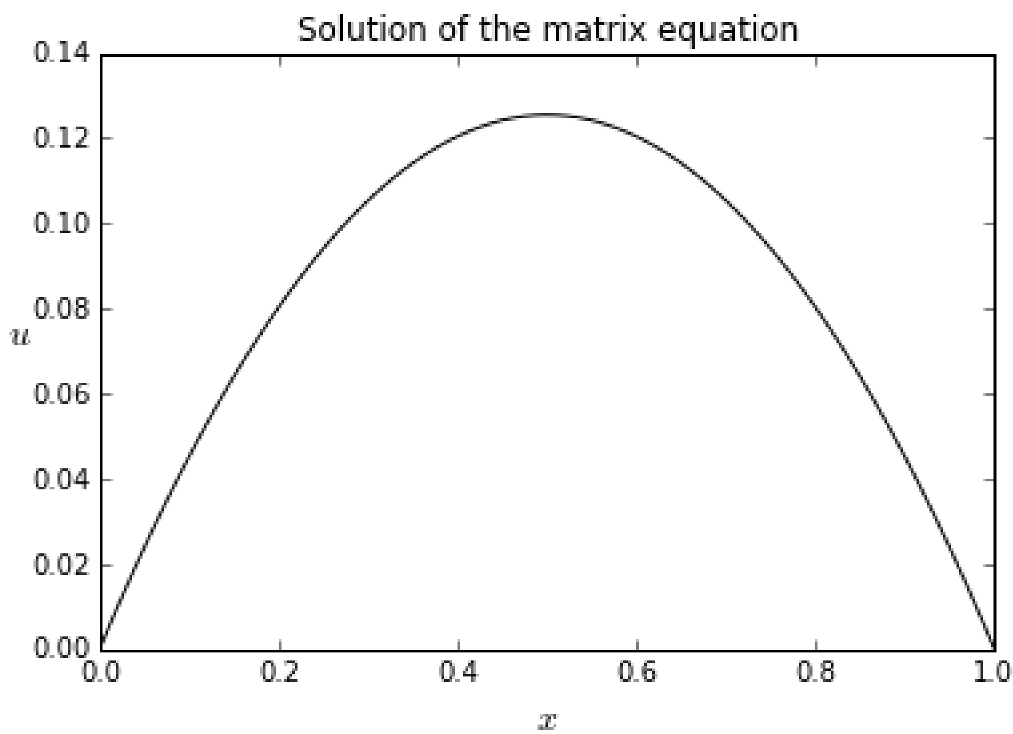
In [8]:
```
N = 1000              # Size of the matrix
dx = 1. / (N - 1)     # Space step size
```

```
x = linspace(0.0, 1.0, N)

# Definition of the tridiagonal matrix
Tmatrix = [ones(N), -2.0 * ones(N), ones(N)]
nonzeropositions = array([-1, 0, 1])
iterationMatrix = dia_matrix(
    (Tmatrix, nonzeropositions), shape=(N, N))

# Schematic representation of the diagonal matrix
figure()
spy(iterationMatrix)
title('Tridiagonal matrix')
draw()

rhs = -ones(N) * dx * dx  # Right-hand side

# Solving the linear system
t = time()
spSolution = spsolve(iterationMatrix, rhs)
comptime = time() - t
print("Computation time using sparse library:  %s seconds" %
      comptime)

# In order to compare with the full resolution
fullIterMatrix = iterationMatrix.todense()
t = time()
linAlgSolution = linalg.solve(fullIterMatrix, rhs)
comptime = time() - t
print("Computation time using standard linear algebra:  %s seconds" %
      comptime)

# Solution plot
figure()
plot(x, spSolution, 'k-')
title('Solution of the matrix equation')
xlabel(u'$x$', fontsize=14)
ylabel(u'$u$', fontsize=14, rotation=0)
show()
```

```
Computation time using sparse library:  0.0011830329895 seconds
Computation time using standard linear algebra:  0.0667409896851 seconds
```



Tridiagonal matrix
0
200
400
600
800
1000



Solution of the matrix equation
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0.00
0.0
0.2
0.4
0.6
0.8
1.0
u
x

3 Example from the Lecture Notes

We consider here the example in the lecture notes, namely the heat equation $\partial_\tau u(\tau, x) = \frac{\sigma^2}{2}\partial_{xx}u(\tau, x)$ on $[0,\infty) \times [0,1]$ with boundary condition $u(0,x) = 2x1_{x\in[0,1/2]} + 2(1-x)1_{x\in[1/2,1]}$.

We first rewrite the explicit scheme as a function, taking the boundary condition as argument.

As seen in the lecture notes, the CFL condition, ensuring convergence of the scheme, is cfl ≤ 1, where cfl $:= \delta_t/\delta_x^2$.

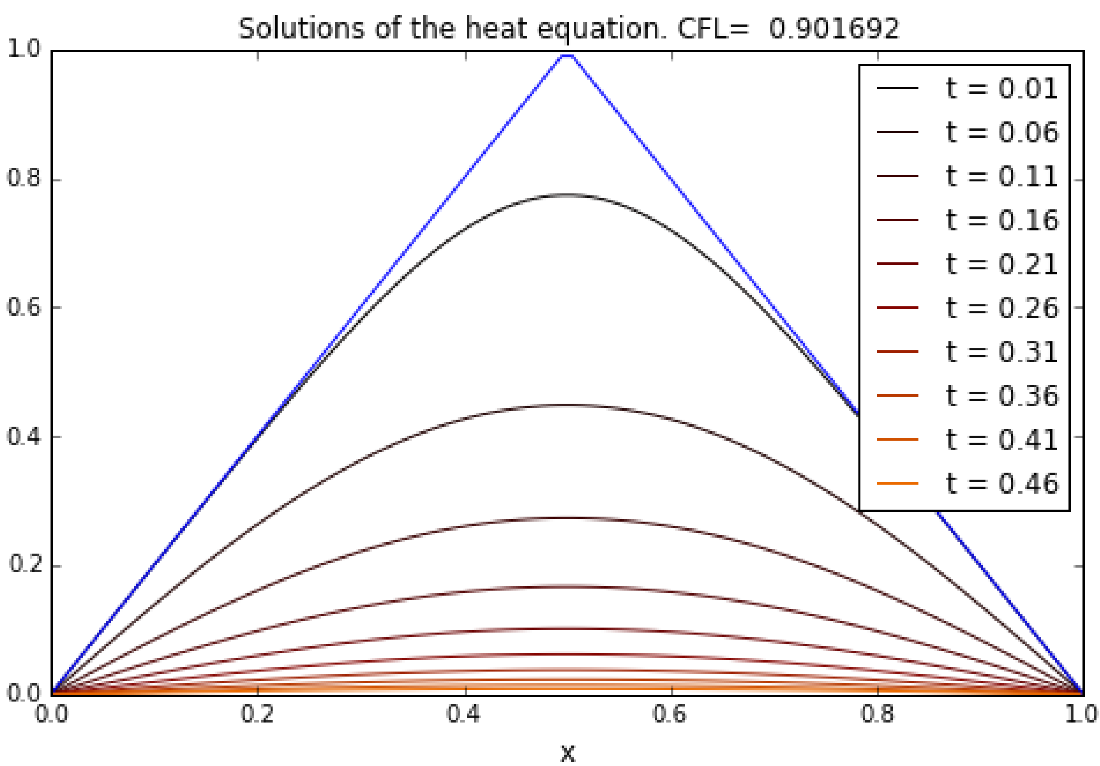
In [9]:
```
def explicitSchemeHeatEquation(sigma, barX, T, m, n, BC, *extraArguments):
    dx = barX / (m - 1)  # Grid step in space
    dt = T / n          # Grid step in time

    # Boundary conditions
    xx = linspace(0., barX, m)
    uu = BC(xx)
    rhs = zeros(m)
    for l in range(0, n):
        rhs[1:-1] = dt * (0.5 * sigma * sigma) * \
            (uu[:-2] - 2.0 * uu[1:-1] + uu[2:]) / (dx * dx)
        uu += rhs
    return xx, uu
```

In [10]:
```
#################################################################
######## Boundary conditions in the example from the lecture note
#################################################################


def boundaryConditionf(zz):
    uu = []
    for z in zz:
        if z < 0.5:
            uu.append(2. * z)
        else:
            uu.append(2. * (1. - z))
    return np.asarray(uu)
#################################################################
#################################################################
```

3.0.1 Convergence of the scheme

In [11]:
```
sigma, barX, T = sqrt(2.), 1., 0.001


nx = 100  # Number of grid points in space
nt = 10000  # Number of time steps
```

In [12]:
```
TT = arange(0.01, 0.5, 0.05)
plt.figure(figsize=(8, 5))

for T in TT:
   xx, uu = explicitSchemeHeatEquation(
       sigma, barX, T, nx, nt, boundaryConditionf)
   plt.plot(
       xx, uu, color=get_cmap('afmhot')(T), label="t = %1.2f" % T)
   plt.legend()

plt.plot(xx, boundaryConditionf(xx))

dx = barX / (nx - 1)
dt = T / nt
cfl = sigma * sigma * dt / (dx * dx)

plt.title("Solutions of the heat equation. CFL=  %s " % cfl)
plt.xlabel(u'x', fontsize=12)
plt.show()
```



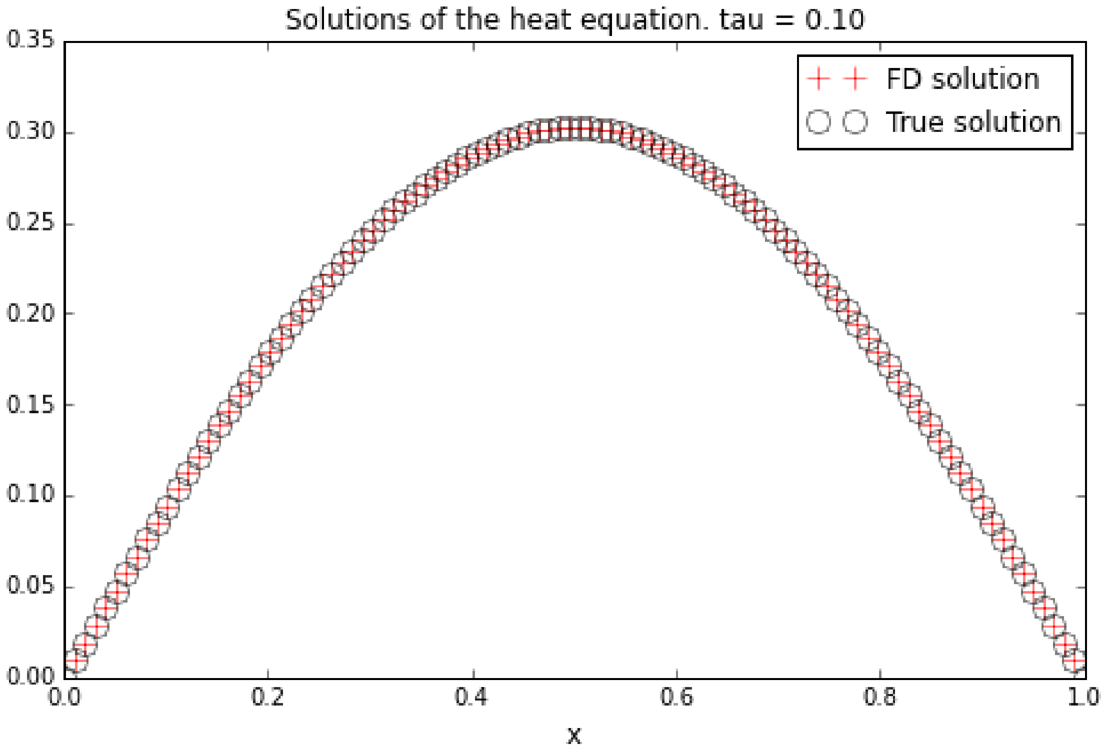
3.0.2 Comparison with the true solution

The true solution has the explicit form:

$$u(\tau, x) = \frac{8}{\pi^2} \sum_{n\geq 1} \frac{\sin(n\pi x)}{n^2} \sin\left(\frac{n\pi}{2}\right) \mathrm{e}^{-n^2\pi^2\tau}.$$

In [13]:
```
def TrueSolution(x, tau, nMax):
    temp = 0.
    for n in range(1, nMax + 1):
        temp = temp + sin(n * pi * x) * sin(0.5 * n *
                                  pi) * exp(-n * n * pi * pi * tau) / (n * n)
        return 8. * temp / (pi * pi)
```

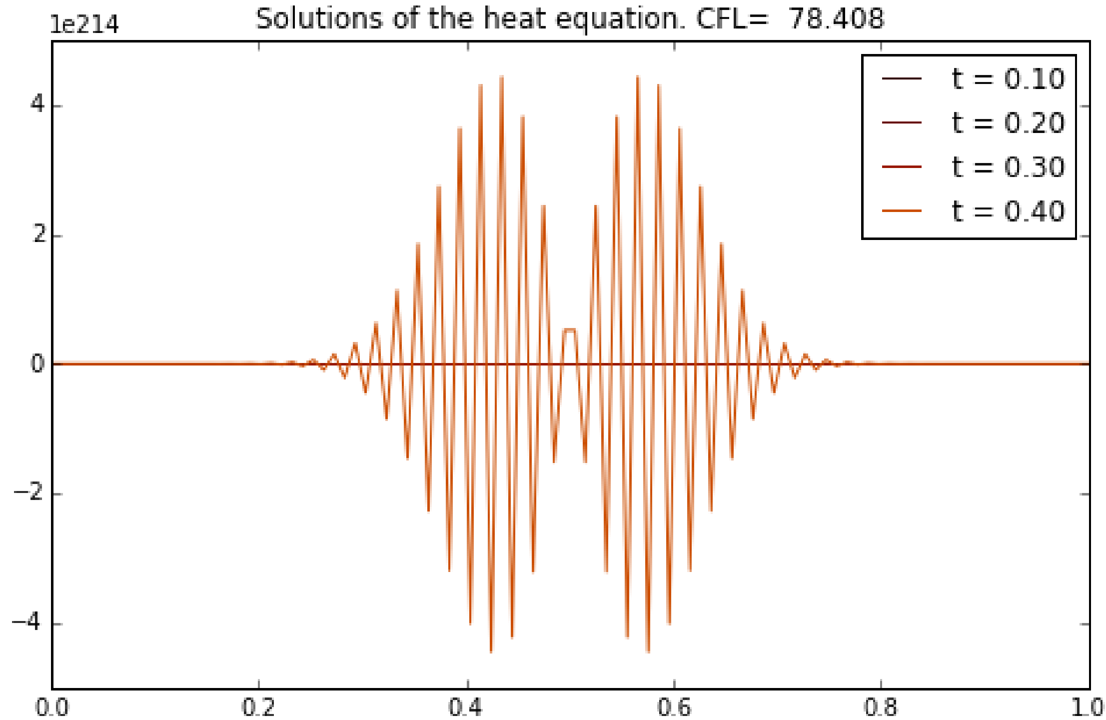
In [14]:
```
nx = 100  # Number of grid points in space
nt = 10000  # Number of time steps

plt.figure(figsize=(8, 5))
tau = 0.1
xx, uu = explicitSchemeHeatEquation(
    sigma, barX, tau, nx, nt, boundaryConditionf)
trueSols = [TrueSolution(x, tau, 20) for x in xx]

plt.plot(xx, uu, 'r+', markersize=10, label="FD solution")
plt.plot(xx, trueSols, 'ro', mfc='none',
         markersize=10, label="True solution")
plt.legend()

plt.title("Solutions of the heat equation. tau = %1.2f" % tau)
plt.xlabel(u'x', fontsize=12)
plt.show()
```



3.0.3 Explosive scheme

```
In [19]: sigma, barX, T = sqrt(2.), 1., 0.001

         nx = 100  # Number of grid points in space
         nt = 100  # Number of time steps

In [20]: TT = arange(0.1, 0.5, 0.1)
         plt.figure(figsize=(8, 5))

         for T in TT:
            xx, uu = explicitSchemeHeatEquation(
                sigma, barX, T, nx, nt, boundaryConditionf)
            plt.plot(
                xx, uu, color=get_cmap('afmhot')(T), label="t = %1.2f" % T)
            plt.legend()

         dx = barX / (nx - 1)
         dt = T / nt
         cfl = sigma * sigma * dt / (dx * dx)
```

```
plt.title("Solutions of the heat equation. CFL=  %s " % cfl)
plt.show()
```




```
In [ ]:
```