AMDP EXAM 2015-2016 – SAMPLE SOLUTIONS

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1. Warm-up: Preliminary questions

(i) [2 points] Stopping time: see the lecture notes for the definition.

(ii) [2 points] Local martingale: Stopping time: see the lecture notes for the definition.

(iii) [2 points] Straightforward computations show that, for any maturity T, the implied volatility smile (as a function of the strike) is constant and equal to $\left(\frac{1}{T}\int_0^T \xi^2(t)\mathrm{d}t\right)^{1/2}$.

(iv) [3 points] Local volatility: see the lecture notes for the definition.

(v) [6 points] The function g is defined via Put-Call parity from the European Put option price. The Put clearly (by dominated convergence) satisfies the standard no-arbitrage bounds $0 \leq \text{Put} \leq (K - S_0)_+$, so that g satisfies the bounds $(S_0 - K)_+ \leq g(K) \leq S_0$. However, the function f does not. Clearly, $f(0) = \mathbb{E}(S_T) < S_0$. By continuity and convexity, there exists K^* such that $f(K) < (S_T - K)_+$ for all $K < K^*$ and $f(K) \geq (S_T - K)_+$ for all $K \geq K^*$. On the interval $[0, K^*)$, the function f therefore does not satisfy the no-arbitrage bounds. In particular, it means that, on this interval, the implied volatility is not well defined.

2. Tail asymptotics of the implied volatility

(i)

$$\limsup_{n\uparrow\infty} u_n = \inf_{n\geq 0}\sup_{m\geq n} u_m \qquad \text{and} \qquad \liminf_{n\uparrow\infty} u_n = \sup_{n\geq 0}\inf_{m\geq n} u_m.$$

Take $u_n := (-1)^n$. Then $\limsup_{n\uparrow\infty} u_n = 1$ and $\liminf_{n\uparrow\infty} u_n = -1$.

(ii) The function ψ is smooth on $(0,\infty)$ with $\lim_{z\downarrow 0}\psi(z) = 2$ and $\lim_{z\uparrow\infty}\psi(z) = 0$.

(iii) In Black-Scholes, we have $\mathbb{E}(S_t^u) = S_0^u \exp\left(\frac{1}{2}u(u-1)\sigma^2 t\right)$. Therefore all moments exist: $p^* = q^* = +\infty$ and hence $\beta_R = \beta_L = 0$. The moment formula implies that the wings of the smile are flat, which is consistent with the fact that the implied volatility smile is actually constant (hence flat) everywhere in the Black-Scholes model.

(iv) (a) The following computation follows directly by integrating $\mathbb{P}(Y_1 \in \mathrm{d}x)$ above:

$$\mathbb{E}\left(\mathrm{e}^{uY_1}\right) = \mathrm{p}\frac{\lambda_+}{\lambda_+ - u} + (1-\mathrm{p})\frac{\lambda_-}{\lambda_- + u}, \qquad \text{for all } u \in \mathcal{D}_Y = (-\lambda_-, \lambda_+).$$

(b) The computation of the Laplace transform of S_t is immediate and follows from the independence of the family (Y_n) and the Poisson process $(N_t)_{t\geq 0}$:

$$\mathbb{E}\left(S_t^u\right) = \exp\left(u\gamma t + \frac{\sigma^2 u^2}{2}t + \lambda t\left\{\mathbb{E}\left(\mathrm{e}^{uY_1}\right) - 1\right\}\right), \qquad \text{for all } u \in \mathcal{D}_Y.$$

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(c) The martingale property holds as soon as $\mathbb{E}(S_t) = 1$, i.e.

$$\gamma + \frac{\sigma^2}{2} + \lambda \left\{ \mathbb{E}\left(\mathrm{e}^{Y_1}\right) - 1 \right\} = 0,$$

from which we deduce the value of γ.

(d) We therefore deduce $q^* = \lambda_-$ and $p^* = \lambda_+ - 1$.

(e) The wings are independent of the time horizon, because S is a Lévy process, with stationary increments. In the Heston model, for example, p^* and q^* depend on T and so do the wings of the smile.

(f) The larger λ_+, the lighter the right tail of the distribution of S_t, since λ_+ represents the intensity of positive jumps. Not surprisingly then, the moment explosion p^* then increases, and, since the function ψ is decreasing, the slope of the total variance in the wings increases. Symmetric arguments hold for the left side of the smile.

3. The OU process

(i)

(a) This follows directly by integration using the transformed process $\widetilde{X}_t := \mathrm{e}^{-\mu t} X_t$.

(b) The coefficients are Lipschitz with bounded linear growth.

(c) Straightforward manipulations show that

$$\int_0^T X_t \mathrm{d}t = \frac{X_T - x - \sigma W_T}{\mu},$$

so that the random variable is Gaussian and its expectation and variance follow by linearity and Itô's isometry.

(ii)

(a) Itô's formula yields

$$\mathrm{d}u(t, W_t) = \left(\partial_t + \frac{1}{2}\partial_{xx}\right) u(t, W_t)\mathrm{d}t + \partial_x u(t, W_t)\mathrm{d}W_t = \partial_x u(t, W_t)\mathrm{d}W_t,$$

so that clearly $(u(t, W_t))_{t \geq 0}$ is a local martingale adapted to the Brownian filtration.

(b) Since any bounded local martingale is also a martingale, $u(\cdot)$ satisfies

$$u(t, W_t) = \mathbb{E}\left(u(T, W_T)|\mathcal{F}_t\right) = \mathbb{E}\left(f(W_T)|\mathcal{F}_t\right) = \mathbb{E}\left(f(W_T - W_t + W_T)|\mathcal{F}_t\right),$$

which yields the result.

(c) With the SDE $\mathrm{d}X_t = X_t \mathrm{d}t + \sigma \mathrm{d}W_t$, then $(u(t, X_t))_{t \geq 0}$ is a local martingale if and only if it satisfies the PDE

$$\left(\partial_t + x\partial_x + \frac{1}{2}\partial_{xx}\right) u = 0,$$

and the solution follows as before.

(iii)

(a) This follows directly from Feynman-Kac.

(b) The solution follows directly from plugging the proposed solution into the PDE and matching the boundary conditions.

4. Strict local martingales

(i)

(a) Direct computation yields
$$\left(\partial_t + \frac{1}{2}S^4\partial_{SS}\right)u(t,S) = 0,$$
for all $(t,S) \in [0,T)\times\mathbb{R}_+$ and with boundary condition $u(T,S) \equiv S$. Therefore the replication strategy is given, as usual, by $\partial_S u(t,S_t)$.

(b) Clearly $(S_t)_{t\geq 0}$ and $(\phi_t)_{t\geq 0}$ are local martingales, and so it $(\Pi_t)_{t\geq 0}$. Since admissible strategies are bounded below (by definition), then, should $(\Pi_t)_{t\geq 0}$ be one, it would therefore be a supermartingale, and in particular $\mathbb{E}(\Pi_T)\ leq\Pi_0 = 0$. However,
$$\mathbb{E}(\Pi_T) = \mathbb{E}(S_0\phi_T - \phi_0 S_T) = (S_0 - \phi_0)\mathbb{E}(S_T).$$
Since $S_0 > \phi_0$, this yields a contradiction.

(ii)

(a) Let $\mathrm{x} = (x_1, x_2, x_3)$. Define $f(\mathrm{x}) := \sqrt{(x_1-1)^2 + x_2^2 + x_3^2}$, so that, $\nabla f(\mathrm{x}) = -f(\mathrm{x})\cdot(x_1 - 1, x_2, x_3)$ and $\|\nabla f(\mathrm{x})\| = 0$. Itô's formula then implies
$$\mathrm{d}X_t = -X_t^4\left\{(Z_t^1 - 1)\mathrm{d}Z_t^1 + Z_t^2\mathrm{d}Z_t^2 + Z_t^3\mathrm{d}Z_t^3\right\},$$
so that X is a local martingale.

(b) This follows by solving the three-dimensional system directly.

(c) By switching to spherical coordinates, we can write
$$\begin{aligned}
\mathbb{E}(X_t) &= \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3_+}\frac{\exp\left(-\frac{x_1^2+x_2^2+x_3^2}{2}\right)}{[(x_1\sqrt{t}-1)^2 + x_2^2 t + x_3^2 t]^{1/2}}\mathrm{dx}\\
&= \frac{1}{(2\pi)^{3/2}}\int_0^\infty\int_0^\pi\int_0^{2\pi}\frac{r^2\sin(\theta)\exp\left(-\frac{r^2}{2}\right)}{(r^2 t - 2\sqrt{t}\cos(\theta)+1)^{1/2}}\mathrm{d}\varphi\mathrm{d}\theta\mathrm{d}r\\
&= \frac{1}{(2\pi)^{1/2}}\int_0^\infty\int_0^{2\pi}\frac{r^2\sin(\theta)\exp\left(-\frac{r^2}{2}\right)}{(r^2 t - 2\sqrt{t}\cos(\theta)+1)^{1/2}}\mathrm{d}\varphi\mathrm{d}\theta\mathrm{d}r\\
&= \frac{1}{(2\pi t)^{1/2}}\int_0^\infty r\exp\left(-\frac{r^2}{2}\right)\sqrt{r^2 t - 2\sqrt{t}\cos(\theta)+1}\mathrm{d}r\Big|_{\theta=0}^{\theta=T}\\
&= \frac{1}{(2\pi t)^{1/2}}\int_0^\infty 2\left(r\mathbb{1}_{\{r?t^{-1/2}\}} + r^2\sqrt{t}\mathbb{1}_{\{r\leq t^{-1/2}\}}\right)\exp\left\{-\frac{r^2}{2}\right\}\mathrm{d}r\\
&= 2\int_0^{t^{-1/2}}\frac{\exp\left(-\frac{r^2}{2}\right)}{\sqrt{2\pi}}\mathrm{d}r = 2\mathcal{N}\left(t^{-1/2}\right) - 1.
\end{aligned}$$
Clearly $\mathbb{E}(X_t) < 1$ and the Put-Call parity is violated for strike equal to zero.

(iii)

(a) Since $\mathbb{P}$ and $\mathbb{Q}^{\mathrm{M}}_{x,t}$ are equivalent, then $\mathbb{P}(Y_t > 0 \text{ for all } t) = \mathbb{Q}^{\mathrm{M}}_{x,t}(X_t > 0 \text{ for all } t) = 1$. Therefore
$$\mathbb{E}^{\mathbb{Q}^{\mathrm{M}}_{x,t}}(X_T|\mathcal{F}_t) = \mathbb{E}^{\mathbb{P}}\left(X_T\,\frac{\mathrm{d}\mathbb{Q}^{\mathrm{M}}_{x,t}}{\mathrm{d}\mathbb{P}}\middle|\mathcal{F}_t\right) = Y_t^{-1} = X_t,$$
which proves the claim.

(b) Itô's formula yields

$$\mathrm{d}X_t = -\frac{\mathrm{d}Y_t}{Y_t^2} + \frac{\mathrm{d}\langle Y\rangle_t^2}{Y_t^3} = -Y_t\sigma_t(\mathrm{d}W_t - \sigma_t\mathrm{d}t).$$

Girsanov's Theorem implies that $\widetilde{W}^{\mathbb{Q}^{\mathrm{M}}_{x,t}} := W - \int \sigma_u \mathrm{d}u$ is a $\mathbb{Q}^{\mathrm{M}}_{x,t}$-Brownian motion, and so is $W^{\mathbb{Q}^{\mathrm{M}}_{x,t}} := -\widetilde{W}^{\mathbb{Q}^{\mathrm{M}}_{x,t}}$

(c) Clearly $\mathrm{d}X_t = X_t\sigma_t\mathrm{d}W_t^{\mathbb{Q}^{\mathrm{M}}_{x,t}} = \mathrm{d}W_t^{\mathbb{Q}^{\mathrm{M}}_{x,t}}$ since Y is a true martingale. Therefore

$$\mathbb{Q}^{\mathrm{M}}_{x,t}(X_t > 0) = \mathbb{Q}^{\mathrm{M}}_{x,t}(W_t > -1) = \mathcal{N}(t^{-1/2}) < 1.$$

Since $\mathbb{P}(X_t > 0) = \mathbb{P}(Y_t > 0) = 1$, the two probabilities $\mathbb{P}$ and $\mathbb{Q}^{\mathrm{M}}_{x,t}$ cannot be equivalent, which concludes the proof.

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