Approximation Behooves Calibration

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Abstract

Calibration based on an expansion approximation for option prices in the Heston stochastic volatility model gives stable, accurate, and fast results for S&P500-index option data over the period 2005 to 2009.
Keywords: Option pricing, expansion, Heston model, calibration, S&P500-index options.

1 Introduction

Our aim with this paper is three-fold. First, to give a short overview of the recent literature on expansion methods for option pricing. This includes an Edgeworth-type formula whose terms are explicit for the Heston (1993) model. We initially thought this was a new result, but then we discovered Sartorelli (2010). The second aim is to test whether the expansion is accurate enough to be used for calibration to real-life data. Several papers, see Guillaume & Schoutens (2010) and the references therein,[1] document that even at the best of of times, calibration of the Heston model is a delicate matter. But with the turmoil in the financial markets since 2007, we are likely to be at the other end of that Dickens novel starter. And maybe the expansion literature just reports benign cases? The title of the paper gives the conclusion. Expansion-based calibration offers approximately a factor five speed-up, is stable, and gives results that are accurate enough to be of practical use. The third aim is to put a good data-set of option prices in the public domain; daily observations of implied volatility surfaces for the S&P500-index over the period 2005-2009 synchronized with the index itself and with estimated complete term structures of interest rates and dividend yields.

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[1]A Google search in March 2013 for 'Heston' + 'calibration' gave $\sim 65,000$ hits.

2 The Heston Model: Transform and Expansion Methods

In the Heston model the risk-neutral price dynamics (that is, dynamics under a pricing/martingale measure $\mathbb{Q}$) of some underlying asset (stock, exchange rate, ...) with dividend yield δ is

$$dS(t) = (r - \delta)S(t)dt + \sqrt{V(t)}S(t)dW_1(t),$$

where r is the (assumed constant) risk-free interest rate and the instantaneous variance follows a square-root process,

$$dV(t) = \kappa(\theta - V(t))dt + \eta\sqrt{V(t)}\left(\rho dW_1(t) + \sqrt{1-\rho^2}dW_2(t)\right),$$

allowing for correlation between changes in underlying and variance. No explicit representation for the density of $\ln S(T)$ is known, but the characteristic function of $\ln S(T)$ (that is, $\mathbf{E}^{\mathbb{Q}}(\exp(iu \ln S(T)))$) can be found in closed form; it is

$$\exp(A(T;u) + B(T;u)v(0) + iu \ln S(0)),$$

where A and B are explicitly known functions that solve Riccati equations.

Our interest is in calculating prices of call-options

$$C(K;T) = e^{-rT}\mathbf{E}^{\mathbb{Q}}((S(T) - K)^+),$$

where T is the expiry-date and K is the strike. Given the characteristic function call-option prices can be found from the numerical calculation of integrals. This gives rise to the so-called transform methods. Immediate use of the inversion theorem gives a double integral. It is simple to rewrite that into two single integrals, and with some sleight of hand it can be reduced it to a single integral, see for instance Lipton (2002), which is our preferred formulation.

However, the numerical implementation may be less straightforward, partly due to a complex logarithm function in the A-function, see Lord & Kahl (2010). Therefore several ways to approximate call-option prices — often w/ the word *expansion* attached — have been proposed. The tools vary, but the idea is usually to boil calculations down to at most Black-Scholes-like terms. Besides avoiding numerical intricacies, advantages of expansion methods are speed improvement, applicability across model-specifications, and possibly improved intuition about the model. One of the first examples is Hagan, Kumar, Lesneiwski & Woodward (2002) who use singular perturbation techniques to derive approximations for options with short expiries in a lognormal stochastic volatility model, a model that has since been acronym'ed SABR. Andreasen & Huge (2010) and Andreasen & Huge (2013) find ordinary differential equations for approximate option prices in models that combine stochastic and local volatility.[2] In Larsson (2012) Malliavin calculus is used derive option price approximations. Drimus (2011) suggests an expansion directly via Greeks and applies it to the Heston model. Later this has been extended to Levy jump models by Nicolato & Sloth (2013). Another approach is based on the so-called Edgeworth expansion which approximates an unknown density function (around, typically, the normal distribution) in terms of its cumulants, these being the coefficients in the power-series expansion of the logarithm of its characteristic function. One of the first financial applications is Aït-Sahalia (1999) who approximates likelihood functions. In Sartorelli (2010) it is used for option pricing, leading to a result that is summarized in the next proposition.

The Sartorelli Approximation *Let $\kappa_1 =: \mu, \kappa_2 =: \sigma^2, \kappa_3, \ldots, \kappa_N$ be the cumulants of the distribution of $\ln S(T)$. The price of an European call-option with strike K and expiry-date T can be approximated by an Edgeworth expansion of order N given by*

$$C_N(K,T) = e^{\mu+\frac{\sigma^2}{2}} P_2^N - K P_1^N,$$

with

$$\begin{aligned}
P_i^N &= Y_0^i + \sigma^1 \left[Y_3^i \frac{S_3}{3!} \right] \\
&+ \sigma^2 \left[Y_4^i \frac{S_4}{4!} + Y_6^i \frac{1}{2!} \left(\frac{S_3}{3!} \right)^2 \right] \\
&+ \sigma^3 \left[Y_5^i \frac{S_5}{5!} + Y_9^i \frac{1}{3!} \left(\frac{S_3}{3!} \right)^3 + Y_7^i \frac{S_3}{3!} \frac{S_4}{4!} \right] \\
&+ \sigma^4 \left[Y_6^i \frac{S_6}{6!} + Y_8^i \frac{1}{2!} \left(\frac{S_4}{4!} \right)^2 + Y_{12}^i \frac{1}{4!} \left(\frac{S_3}{3!} \right)^4 + Y_8^i \frac{S_3}{3!} \frac{S_5}{5!} + Y_{10}^i \frac{1}{2!} \left(\frac{S_3}{3!} \right)^2 \frac{S_4}{4!} \right] \\
&+ \cdots \\
&+ \sigma^N \left[\sum_{\{\zeta_N\}} Y_{N+2\Sigma(\zeta_N)}^i \prod_{1 \le m \le N,\, l_m > 0} \frac{1}{l_m!} \left(\frac{S_{m+2}}{(m+2)!} \right)^{l_m} \right],
\end{aligned}$$

[2] Andreasen & Huge (2013) also give a derivation of a simplified version of the Hagan et al. (2002) formula — "light SABR" as it were.

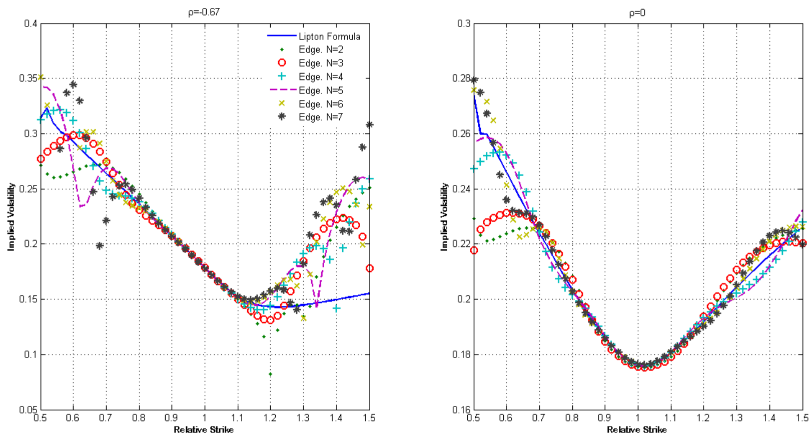

Figure 1: Expansion approximations for Bakshi et al. (1997) Heston parameters with two choices of correlation (left panel: actual estimate, right panel: 0). Option prices are expressed in terms of implied volatilities.

where $\{\zeta_s\} = \{(l_1, \cdots, l_s) \in \mathbb{N}_0^s : l_1 + 2l_2 + \cdots s l_s = s\}$, $\Sigma(\zeta_s) = l_1 + \cdots + l_s$, $S_m = \kappa_m/\sigma^{2(m-1)}$, $Y_j^i \equiv Y_j\left(\frac{\ln K - \mu}{\sigma}, \sigma(i-1)\right)$. *The* Y_n*-functions are determined recursively by*

$$
\begin{aligned}
Y_0(x,y) &= 1 - \Phi(x-y), \\
Y_n(x,y) &= yY_{n-1}(x,y) + \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-y)^2}{2}} H_{n-1}(x)
\end{aligned}
$$

where the H_n*'s are the Hermite polynomials and* Φ *is standard normal distribution function. For the Heston model, where the characterstic function is known, the cumulants of* $\ln S(T)$ *can be found explicitly, but the length of the expressions grow exponentially.*

The Sartorelli[3] approximation can be applied to any model where the cumulants are known — various Levy jump models for instance — but it might not be very accurate.[4] Figure 1 shows approximate and "true" (Gauss-Lobatto integration of the formula from Lipton (2002)) option prices for Heston model parameters (mostly) given in Bakshi, Cao & Chen (1997). In our Matlab-implementation we can calculate about 1,000 prices per second. (As a ball-park figure a direct implementation of the

[3]The name is actually apt. In the same mnemonic way that a Taylor expansion is tailor-made, this could be called a sartorial approximation.

[4]This is the subject of ongoing research.

Black-Scholes formula produces $\sim 66,000$ prices/sec.) The 2nd order expansion is about 100 times faster than Heston integration, the 7th order is about three times faster. The graph shows what is in our experience characteristic features of expansion methods. They work well at-the-money ($\pm$ 20% moneyness, roughly), but deteriorate in quality for out-of-the-money options as well as with increasing absolute correlation. Adding more terms in the expansion may help, but certainly it does not do so in a uniform or monotone way. This also hints at why rigorous convergence statements are largely absent from the option price expansion literature. The approximations are typically not even arbitrage-free, which in the case without dividends means convex in strike and increasing in expiry, although Doust (2012) provides a way to remedy this.

Apriori there are, thus, pros and cons of using expansion methods for calibration to real-life data. Speed is important and the moneyness of the target instruments is not extreme, but equity markets typically display strong negative correlation between underlying and volatility. So we are set up nicely for a horse-race.

3 Data

As testing ground for the calibrations we use data on European type options on the S&P500-index period from early 2005 to mid-2009. Most noticeably, see Figure 2, this period includes the financial turmoil that started in the Summer of 2007 with the Subprime Crisis, took off dramatically with the Lehman Brothers' default in September 2008, and still reverberates to this day. We have daily data on option prices; 182 implied volatilities for each day in our data-set with moneyness from -30% to +30% (of current underlying) and between one month and three years to expiry. The data-set contains synchronous information about the price of the underlying and term structures of interest rates and dividend yields. The data have been provided for use in research by a major investment bank (it is the volatility surfaces that the bank itself uses) with a time-delay (therefore, updating data to, say, 2013 is not straightforward) and under the condition that the bank remains nameless. Files with the data (`Data[1/2].xls`) can be found at `http://www.math.ku.dk/~rolf/Svend/`.

4 Empirical Results

To calibrate the model we chose on any given day, t, the vector $\vartheta = (\theta, \kappa, \sigma, \rho, V_t)$ (strictly speaking this is a mixture of parameters and one state variable; we will refer

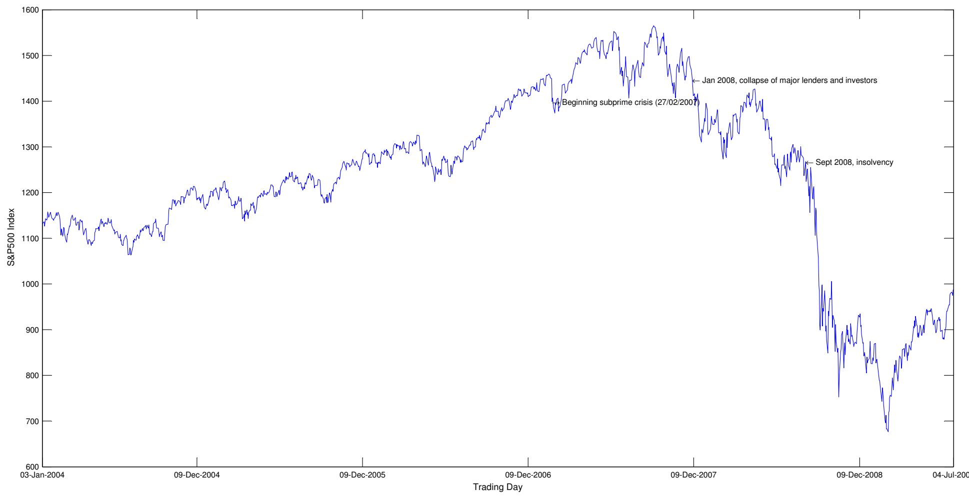

Figure 2: The S&P500-index 2004 to 2009.

to ϑ as a parameter anyway) that solves

$$\min_{\vartheta} \sum_i (C_{obs}(K_i, T_i) - C_{model}(K_i, T_i; \vartheta))^2,$$

where the summation runs over that day's observed option prices across strikes and expiries. Other goodness-of-fit measures can be used (absolute vales or root-mean-square rather than sum of squares, implied volatilities rather prices, relative differences); we found that using these lead to similar results as reported in the following. Guillaume & Schoutens (2010) suggest mixing cross-sectional and time-series information. We have not implemented this.

Figure 3 shows the results of the calibrations. The panels depict (reading from left to right and to to bottom) estimates of η, κ, $\sqrt{V_t}$, θ, and ρ. The bottom right panel shows the computation speed measured as no. calibrations per minute. Each calibration comes in six versions: 2nd and 4th order expansion and "true" $\otimes$ with and without the Feller condition (which we will discuss shortly). We see that for the estimates from the 4th order expansion without the Feller condition differences to the "true" Heston estimates are small. It is only for the correlation parameter (note that these estimates are "very negative", below -0.8) and for the vol-of-vol (η) after 2008 that differences are visible to the naked eye. And the expansions are faster then "true" Heston; including the optimization the 4th order expansion performs about five times more calibrations per minute.

We end the paper by discussing the Feller condition and the effect of the post-2007 financial turmoil.

The Feller condition If $2\kappa\theta \geq \eta^2$, then the instantaneous variance process V stays strictly positive.This is known as the Feller condition.[5] However, if the process does hit 0, it is immediately reflected back into positive range (this is the implication of using of "usual formulas"), and the discounted price of the underlying is still a martingale. Some sources in the literature do impose the Feller condition, but this appears not to be market practice. As one market participant told us: "I would never impose the Feller condition — it is for people who believe in elves and fairies".[6] So, nothing breaks down. Nor does imposing the Feller condition remove problems with simulation near zero that are analyzed in Lord, Koekkoek & van Dijk (2010). We ran the empirical experiment with and without the Feller condition. In the unrestricted case, the parameter estimates violated the Feller condition at almost every date. The goodness-of-fit without the Feller condition is, as it should be, better; this quite visible for sum of squares (which is what is being minimized) and root-mean-square error, and holds to a lesser extend for the "digital" criterion of "hitting the bid/ask-spread" (discussed next and shown in Figure 4). One odd feature in Figure 3 is that the place where Feller-effect most clearly manifests itself is in the correlation estimate — despite the correlation parameter not entering the condition.

The financial turmoil Figure 3 shows that estimates are affected by the financial turmoil; it begins in the Summer of 2007 and the effect is strong from Lehman Brothers' default in September 2008. Changes in the instantaneous variance are of course consistent with a stochastic volatility model.[7] But changes in parameters, strictly logically, are not, and should be viewed as evidence against the model, although this level of fundamentalism will not get you far in financial modelling. Another way to see the deteriorating model quality is to look at percentage of the calibrated model prices that fill within the bid/ask-spreads. The input-data do not contain explicit information about bid/ask-spreads, but according to Wystup (2007) a multiplicative spread on volatilities of 2% is common for at-the-money options in the Interbank market. Or in numerical terms: a typical at-the-money option is sold at 0.204, bought 0.196. In Figure 4 we use (conservatively) the 2%-spread across moneyness. With the financial turmoil we go from a situation where about 85% of the prices generated by the calibrated models fall within bid/ask-spreads down to only 50% from late 2008

[5] More accurately, the version with strict inequality is. The validity of our previous statement about the "="-case can be proved by applying Feller's test for explosions to $\ln V$.

[6] He then added that he might want to impose other conditions to avoid moments explosions as discussed in Andersen & Piterbarg (2007).

[7] Arguably *such* strong changes are not *very* consistent with *this particular* model, but that analysis is for another paper.

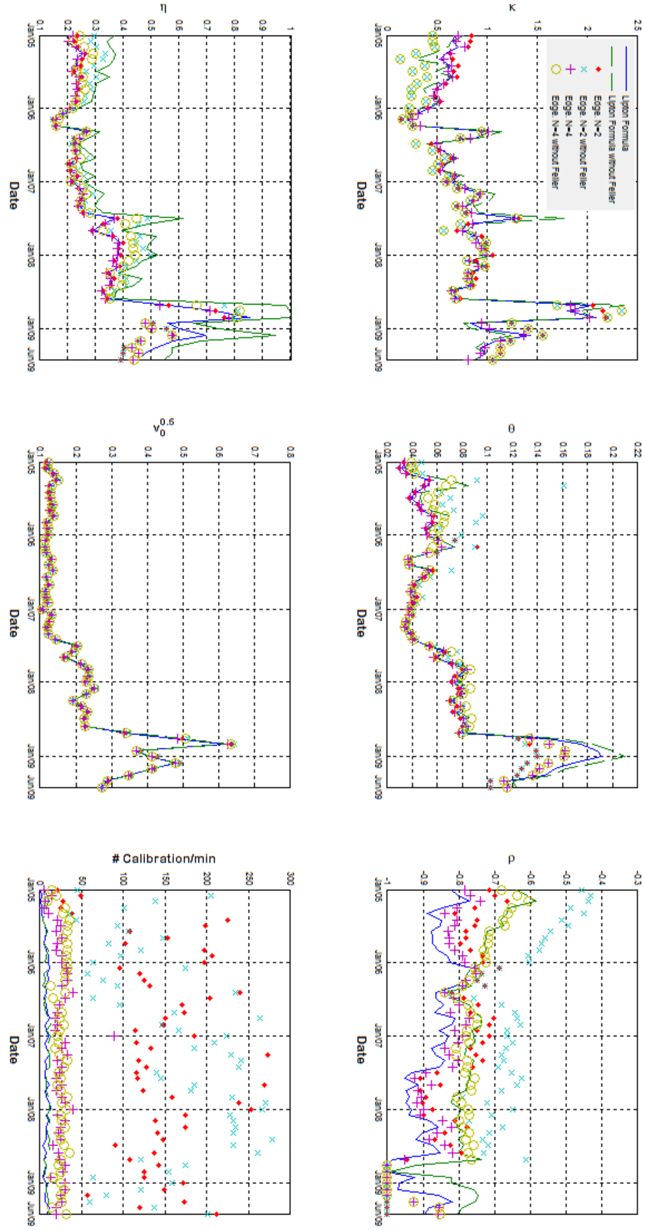

Figure 3: Parameter estimates when the Heston model is calibrated to S&P500-index options 2005-2009.

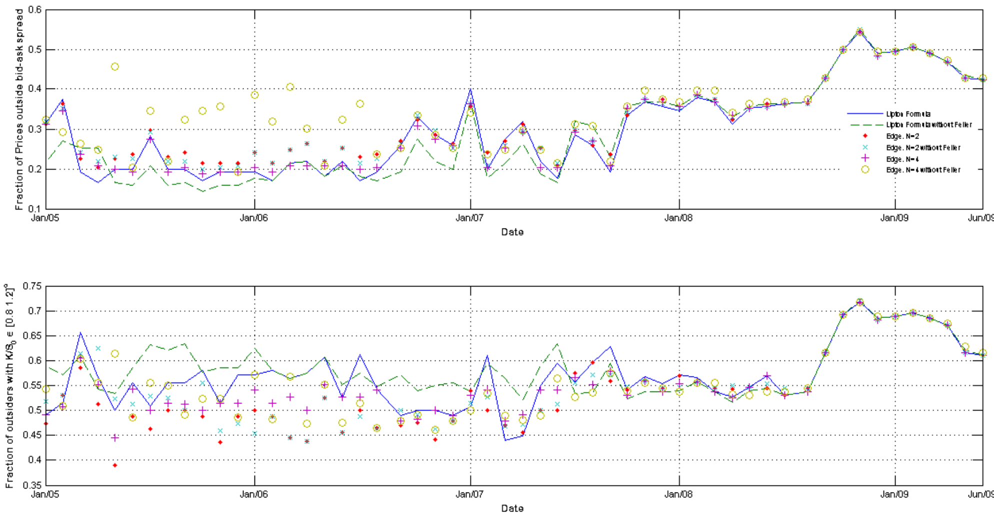

Figure 4: Fraction of calibrated model prices that fall within the bid/ask-spread. The top panel includes all options, the bottom panel options with absolute moneyness above 20%.

and on. The bottom panel shows that the errors stem mostly from options with absolute moneyness above 20%.

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