No-Arbitrage Bounds on Two One-Touch Options*

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Abstract

This paper investigates the pricing bounds of two one-touch options with the same maturity but different barrier levels, where the pricing bound is a range within which a one-touch option can take a price when a price of another one-touch option is given. The upper or lower bounds are the cost of a super-replicating portfolio and a sub-replicating portfolio respectively. These consist of call options, put options, digital options and a one-touch option. We assume that the underlying process is a continuous martingale, but do not postulate a model.

Keywords: barrier option, one-touch option, model-independent, super-replication, pricing bounds
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1 Introduction

This paper investigates pricing bounds within which a one-touch option can take a price when the price of another one-touch option with the same maturity but a different barrier level is given.

Financial markets trade many barrier option types such as single/double barrier knock-in/-out options. Of these, one-touch and no-touch options are the simplest barrier options and widely are traded. A one-touch option is a barrier option that pays a unit of currency at the maturity if the barrier is hit and is worthless if the barrier has not been hit. In contrast, a no-touch option is worthless if the barrier is hit. These are important instruments for traders of barrier options, because they reflect a market view of the probability of the barrier being hit.

There has been considerable research on pricing and hedging barrier options. In particular, researchers have proposed several methods that semi-statically hedge barrier options (see e.g. Carr and Chou (1997), Carr et al. (1998) and Derman et al. (1995)). Here, semi-static hedging means the replication of barrier options by trading European puts and calls no more than once after inception. Hedging strategies require options, thus models that price barrier options must be calibrated to these. However, even if the model is perfectly calibrated to a volatility surface there are risks attached to the valuation of barrier options. For instance, Hirsa et al. (2003), Lipton and McGhee (2002) and Schoutens et al. (2005) all state that although models may produce similar European put and call option prices, they give markedly different barrier option

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prices. Touch options are recognized as important products because they are used as an instrument to which a model is calibrated (see e.g. Carr and Crosby (2010)).

The model-independent approach has also been considered for exotic derivatives including barrier options (see e.g. Hobson (1998), Hobson and Neuberger (2012), Labordére et al. (2012) and Hobson and Klimmek (2012)). In particular, Brown et al. (2001) propose robust super-replicating and sub-replicating barrier option strategies including touch-options without assuming any specific models. Cox and Obłój (2011a) and Cox and Obłój (2011b) focus on touch options with two barrier levels in the same manner as Brown et al. (2001). They use call options and put options as well as digital options with the same maturity as replicating instruments and trade forward contracts at the first barrier(s) hitting time(s). Generally, pricing bounds derived from model-independent replications tend to be rather wide, which is also the case for touch-options. Hence, it is worth investigating how much these pricing bounds are refined if other instruments are traded.

This paper investigates pricing bounds within which a one-touch option can take a price when a price of another one-touch option with the same maturity but a different barrier level is given and those European options (including call, put and digital options) with the same maturity. Suppose there is a pricing operator on European options with a certain maturity and a touch option with the same maturity and a certain barrier level. The question is how to extend this pricing operator to a space spanned by a touch option with the same maturity but a different barrier level as well as these derivatives. To address this, we propose pricing operators that provide upper and lower bounds for the touch option based on a super-replication and a sub-replication. Our approach is in line with Brown et al. (2001), Cox and Obłój (2011a) and Cox and Obłój (2011b), in that we assume the underlying asset process is a continuous martingale and our replications consist of static portfolios and transactions of a forward contract in the first instances of hitting the barrier levels. We differentiate by using a touch option as well as European options for the static portfolios.

Moreover, we provide pricing bounds on a touch/no-touch option that pays one unit of currency if and only if the first barrier is hit but the second is not. In Section 4, we consider the pricing bounds on this touch/no-touch option using the one-touch option with the second barrier as well as European options. If we use, instead of the one-touch option, the upper- or lower bounds and the super- or sub-replications on this, we obtain the pricing bounds as well as super- and sub-replications of the touch/no-touch option using only European options.

The next section of this paper describes the settings and notations. The third section reviews the research of Brown et al. (2001). The super-replications and sub-replications for a one-touch option using another with a different barrier level are derived in the fourth section. The fifth section provides numerical examples.

2 Settings and Notations

The settings and notations used in this paper are stated here.

First, we introduce some notations. Let us denote the spot price of the underlying asset at time $t \in [0, T^*]$ by S_t, where T^* is some arbitrary time horizon and the time-t price of a call option and a put option with strike K and maturity $T \in [0, T^*)$ by $C_t(K)$ and $P_t(K)$ respectively. The one-touch option is assumed to be a single knock-in option with maturity T and barrier level $B \in (S_0, +\infty)$. This option is worthless if B has not been hit by the expiration date. If the barrier is hit at any time during the option's life, the terminal payoff is 1. Then, the payoff of the barrier option is $1_{\{\tau_B \le T\}}$, where τ_B is the first time of hitting B:

$$\begin{aligned} \tau_B &:= \tau_B(S) \\ &:= \inf\{t < T^* \mid S_t \ge B\}. \end{aligned} \tag{2.1}$$

A time-t price of this option is denoted as $O_t(B)$. The subscript t may be omitted in case of $t = 0$ such as $C(K)$, $P(K)$ and $O(B)$ for simplicity.

Second, we make some assumptions. The first assumption is that the underlying price process S is a non-negative martingale. The interest rates are also assumed to be zero. This is merely for simplicity, since our results are valid by reading all prices of all options and portfolios as forward T prices in case of a non-zero interest rate. Examples to which our results are applied are that the underlying process is a forward price or that the underlying asset pays continuous dividends equal to the interest rate. We assume that forward transactions are costless and all instruments — such as underlying asset, forward, — are traded without transaction costs. Importantly, we assume that the underlying price process is continuous. This allows us to exchange a call option with strike K, with $(B-K)$ amounts of cash and a put option with the same strike by trading a forward contract with zero cost at the first time of hitting B, since the following parity holds:

$$C_{\tau_B}(K) - P_{\tau_B}(K) = B - K. \tag{2.2}$$

This type of trade is used throughout this paper. Moreover, we add an assumption in Section 4.1 and 4.2 that the distribution of the underlying asset at maturity T under a risk-neutral measure is given. This distribution is centered at S_0. We consider the case where only a finite number of call options are known in Section 4.3. Knowledge of the distribution is equivalent to the knowledge of European call option prices without arbitrage opportunities for the continuum of strikes by Breeden and Litzenberger (1978). The conditions for no arbitrage are well-documented in Davis and Hobson (2007). We assume $C(B) > 0$ to avoid a trivial case. We denote by ν the risk-neutral distribution of the spot price at maturity T determined by prices of these options. It is also assumed that call options, put options and digital call options can be used as replication, where the digital call option with strike K is an option whose payoff is $1_{\{K \le S_T\}}$ in this paper.

Third, we state the aim of this paper: to extend a pricing operator φ that is a linear operator defined on $\mathcal{X} := \mathcal{L}^1([0,+\infty), \nu)$, a set of Lebesgue integrable functions on $[0,+\infty)$ with respect to ν, which associates a payoff of an European option with its initial price such as $\varphi(K) = K$, $\varphi((S_T - K)_+) = C(K)$, $\varphi((K - S_T)_+) = P(K)$. If a price of a one-touch option whose payoff is $1_{\{\tau_B \le T\}}$ is known, we can extend the operator φ to $\mathcal{X} \oplus \mathcal{Y}$, where $\mathcal{Y}$ is a linear space spanned by $1_{\{\tau_B \le T\}}$ and $\oplus$ means a direct sum. This paper examines how to extend the operator φ to $\mathcal{X} \oplus \mathcal{Y} \oplus \tilde{\mathcal{Y}}$, where $\tilde{\mathcal{Y}}$ is a linear space spanned by $1_{\{\tau_{\tilde{B}} \le T\}}$ with another barrier level $\tilde{B}$. To address this, we propose sharp pricing bounds on one-touch options and the corresponding replicating strategies, where *sharpness* means that the pricing bounds can not be improved without adding any other assumption. The lower and upper bounds on the option are defined as follows under our settings:

$$W^L := \inf_{\mathcal{P}} \mathbb{E}\left[1_{\{\tau_{\tilde{B}}(S) \le T\}}\right] \tag{2.3}$$

$$W^G := \sup_{\mathcal{P}} \mathbb{E}\left[1_{\{\tau_{\tilde{B}}(S) \le T\}}\right], \tag{2.4}$$

where $\mathcal{P}$ is a set of all risk-neutral probability spaces $(\Omega, \mathcal{F}, \mathbb{Q})$ and a continuous martingale process $\{S_t\}_{t \in [0,T^*]}$ on it that satisfies $\varphi(\cdot) = \mathbb{E}[\cdot]$ on $\mathcal{X} \oplus \mathcal{Y}$ and $\mathbb{E}$ is an expectation operator corresponding to the probability space. Prices of super-replicating and sub-replicating portfolios are superior and inferior, but not necessarily equal, to W^G and W^L respectively. To prove the sharpness, we find super-replicating and sub-replicating portfolios whose prices are equal to $\mathbb{E}\left[1_{\{\tau_{\tilde{B}}(S) \le T\}}\right]$ with respect to a certain element of $\mathcal{P}$.

Finally, we introduce some further technical notations. Every function f considered in this paper is a combination of the call price function C. We expand the domain of the function f from $[0,+\infty)$ to $\mathbb{R}$ by $C(K) := C(0) - K$ for $K < 0$ (recall that we assume that the underlying process is non-negative). The function has left- and right-sided directional derivatives as does the function C. In this paper, we denote ∂_K^- as the left-sided derivative operator. Moreover, the derivatives have finite total variations and the derivative ∂_{KK}^- can be defined except for a countable set. The subdifferential of a function f at K can be defined and is denoted by

$$\partial_K f(K) := \{k \in \mathbb{R} \mid f(\kappa) \ge f(K) + k(\kappa - K),\ \forall \kappa \in \mathbb{R}\}. \tag{2.5}$$

We introduce the following notation for simplicity:

$$\mathcal{N}(\partial_K f) := \{K \in \mathbb{R} \mid 0 \in \partial_K f(K)\}. \tag{2.6}$$

3 Review of Brown et al. (2001)

In this section, we review the replications for a one-touch option with only European options, as proposed by Brown et al. (2001), because we use these results in Section 4. The one-touch option is assumed to have a barrier level B, where $S_0 < B$.

First, we prepare the following lemma:

Lemma 1. *Suppose that there is a measurable set $\Omega_0 \in \mathcal{F}$ such that $S_T \in [B, +\infty)$ on Ω_0 and $\mathbb{E}[S_T : \Omega_0] = B\mathbb{Q}[\Omega_0]$. Then, there exists a continuous martingale $\{S_t^*\}_{t\in[0,T]}$ such that $S_T = S_T^*$ and $\mathbb{Q}[\tau_B(S^*) \leq T] = \mathbb{Q}[\Omega_0]$.*

Proof. Let X_0, X_1 and X_2 be random variables defined as $X_0 := S_0$, $X_2 := S_T$ and

$$X_1 := B \cdot 1_{\Omega_0} + \beta \cdot 1_{\Omega_0^c}, \tag{3.1}$$

where

$$\beta := B - \frac{B - S_0}{\mathbb{Q}[\Omega_0^c]}. \tag{3.2}$$

Note that $\beta < B$ and $\mathbb{E}[S_T : \Omega_0^c] = \beta\mathbb{Q}[\Omega_0^c]$. Then, $\{X_n\}_{n=0,1,2}$ is a discrete martingale with respect to a filtration generated by X. By Dudley's theorem (see, for instance, p.188 of Karatzas and Shreve (1988)), the random variables $X_1 - X_0$, $(X_2 - X_1) \cdot 1_{\Omega_0}$ and $(X_2 - X_1) \cdot 1_{\Omega_0^c}$ can be expressed with stochastic integrals with respect to the Winner processes. A continuous martingale process S_t^* such that $\mathbb{Q}[\tau_B(S^*) \leq T] = \mathbb{Q}[\Omega_0]$ can be constructed by these stochastic integrals. □

3.1 Super-Replication

Consider the following self-financing strategy $\mathcal{G}(K;B)$ for $\forall K \in [0, B)$:

1. At the initial outset
   - Buy $\frac{1}{B-K}$ units of a call option with strike K.
2. At the first time of hitting B
   - Sell $\frac{1}{B-K}$ units of the forward contract.

The strategy $\mathcal{G}(K;B)$ super-replicates the one-touch option with any $K \in [0, B)$. We provide some optimal strategies properties.

Definition 1. *The initial value of strategy $\mathcal{G}(K;B)$ is defined as*

$$G(K;B) := \frac{C(K)}{B - K}, \tag{3.3}$$

$G_*(B)$ as the infimum value of $G(K;B)$ with respect to K, $K_G(B)$ as a strike price by which the infimum is attained:

$$\begin{aligned} G_*(B) &:= \inf_{K\in(-\infty,B)} G(K;B) \\ &= G(K_G(B);B) \end{aligned} \tag{3.4}$$

and $\mathcal{G}_*(B)$ as the corresponding strategy.

Proposition 1. *The infimum of Eq.(3.4) is attained by any element of* $\mathcal{N}(\partial_K G(B))$*, an interval of* $[0, B)$*. For all* $K_G \in \mathcal{N}(\partial_K G(B))$*, the following holds:*

$$\mathbb{Q}[K_+ < S_T] \leq G_*(B) = \mathbb{E}\left[\frac{S_T - K_G}{B - K_G} : K_G \leq S_T\right] \leq \mathbb{Q}[K_- \leq S_T], \tag{3.5}$$

where $K_- := \inf \mathcal{N}(\partial_K G(B))$ *and* $K_+ := \sup \mathcal{N}(\partial_K G(B))$*. In addition, there is a continuous martingale process* $\{S_t^G\}_{t\in[0,T]}$ *such that*

$$G_*(B) = \mathbb{Q}\left[\tau_B\left(S^G\right) \leq T\right]. \tag{3.6}$$

Proof. By differentiating G with respect to K, we obtain

$$\begin{aligned} \partial_K^- G(K) &= \frac{1}{B-K}\left(\partial_K^- C(K) + \frac{C(K)}{B-K}\right) \\ &= \frac{1}{B-K}\left(\partial_K^- C(K) + G(K)\right) \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} \partial_{KK}^- G(K) &= \frac{1}{B-K}\partial_{KK}^- C(K) + \frac{2}{(B-K)^2}\partial_K^- C(K) + 2\frac{C(K)}{(B-K)^3} \\ &= \frac{1}{B-K}\partial_{KK}^- C(K) + \frac{2}{B-K}\partial_K^- G(K). \end{aligned} \tag{3.8}$$

Since $\partial_K^- G(0) = \frac{1}{B}(-1 + \frac{S}{B}) < 0$, $\lim_{K\to B} \partial_K^- G(K) = +\infty$ and because $\partial_{KK}^- G > 0$ if $\partial_K^- G > 0$, the set $\mathcal{N}(\partial_K G(B))$ is an interval of $[0, B)$ and we have Eq.(3.5). Apply Lemma 1 with $\Omega_0 \subseteq \Omega$ such that $\{K_G < S_T\} \subseteq \Omega_0 \subseteq \{K_G \leq S_T\}$ and $\mathbb{Q}[\Omega_0] = G_*$, then we have a continuous martingale $\{S_t^G\}_{t\in[0,T]}$. □

3.2 Sub-Replication

Consider the following self-financing strategy $\mathcal{L}(K; B)$ for $\forall K \in [0, B)$:

1. At the initial outset
   - Buy $\frac{1}{B-K}$ units of a call option with strike B.
   - Buy 1 unit of a digital call option with strike B.
   - Sell $\frac{1}{B-K}$ units of a put option with strike K.
2. At the first time of hitting B
   - Sell $\frac{1}{B-K}$ units of the forward contract.

The strategy $\mathcal{L}(K; B)$ super-replicates the one-touch option with any $K \in [0, B)$. We provide some optimal strategies properties.

Definition 2. *The initial value of the strategy* $\mathcal{L}(K; B)$ *is defined as*

$$L(K; B) := \frac{C(B)}{B-K} - \frac{P(K)}{B-K} - \partial_K^- C(B), \tag{3.9}$$

$L_*(B)$ *as the supremum value of* $L(K; B)$ *with respect to* K*,* $K_L(B)$ *as a strike price by which the supremum is attained:*

$$\begin{aligned} L_*(B) &:= \sup_{K\in(-\infty,B)} L(K; B) \\ &= L(K_L; B), \end{aligned} \tag{3.10}$$

and $\mathcal{L}_*(B)$ *as the corresponding strategy.*

Proposition 2. *The supremum of Eq.(3.10) is attained by any element of $\mathcal{N}(\partial_K L(B))$, an interval of $[0, B)$. For all $K_L \in \mathcal{N}(\partial_K L(B))$, the following holds:*

$$\mathbb{Q}\left[S_T < K_-, B \leq S_T\right] \leq L_*(B) = \mathbb{E}\left[\frac{S_T - K_L}{B - K_L} : S_T \leq K_L, B \leq S_T\right] \leq \mathbb{Q}\left[S_T \leq K_+, B \leq S_T\right], \quad (3.11)$$

where $K_- := \inf \mathcal{N}(\partial_K L(B))$ and $K_+ := \sup \mathcal{N}(\partial_K L(B))$. In addition, there is a continuous martingale process $\left\{S_t^L\right\}_{t\in[0,T]}$ such that

$$L_*(B) = \mathbb{Q}\left[\tau_B\left(S^L\right) \leq T\right]. \quad (3.12)$$

Proof. By differentiating L with respect to K, we obtain

$$\begin{aligned} \partial_K^- L(K) &= \frac{1}{B-K}\left(\frac{C(B)}{B-K} - \partial_K^- P(K) - \frac{P(K)}{B-K}\right) \\ &= \frac{1}{B-K}\left(L(K) + \partial_K^- C(B) - \partial_K^- P(K)\right) \end{aligned} \quad (3.13)$$

and

$$\begin{aligned} \partial_{KK}^- L(K) &= 2\frac{C(B)}{(B-K)^3} - \frac{1}{B-K}\partial_{KK}^- P(K) - 2\frac{1}{(B-K)^2}\partial_K^- P(K) - 2\frac{P(K)}{(B-K)^3} \\ &= \frac{2}{B-K}\partial_K^- L(K) - \frac{1}{B-K}\partial_{KK}^- P(K). \end{aligned} \quad (3.14)$$

Since $\partial_K^- L(0) = \frac{S}{B^2} > 0$, $\lim_{K\to B} \partial_K^- L(K) = -\infty$ and because $\partial_{KK}^- L < 0$ if $\partial_K^- L < 0$, the set $\mathcal{N}(\partial_K L(B))$ is an interval of $[0, B)$ and we have Eq.(3.11). Apply Lemma 1 with $\Omega_0 \subseteq \Omega$ such that $\{K_L < S_T, B \leq S_T\} \subseteq \Omega_0 \subseteq \{K_L \leq S_T, B \leq S_T\}$ and $\mathbb{Q}[\Omega_0] = L_*$, then we have a continuous martingale $\left\{S_t^L\right\}_{t\in[0,T]}$. ☐

4 Replication using another One-Touch Option

Here, we consider super-replication and sub-replication for a one-touch option with a barrier level B_1 using European options and a one-touch option with a barrier level B_2, where $S_0 < B_1 < B_2$. Rather than considering the barrier option, we consider a touch/no-touch option whose payoff is $1_{\{\tau_1 \leq T < \tau_2\}}$ where τ_1 and τ_2 are the first times of hitting B_1 and B_2 respectively, because of $1_{\{\tau_1 \leq T < \tau_2\}} = 1_{\{\tau_1 \leq T\}} - 1_{\{\tau_2 \leq T\}}$.

For easing expression, we introduce the notation $\pi : [0, 1] \longrightarrow \mathcal{F}$: where $\pi(p)$ is an element of $\mathcal{F}$ for $p \in [0, 1]$ such that $\mathbb{Q}[\pi(p)] = p$, and $S_T(\omega) \leq S_T(\omega^c)$ for $\omega \in \pi(p)$ and $\omega^c \notin \pi(p)$. We also define $\pi([p, q)) := \pi(p)^c \cap \pi(q)$ for $p, q \in [0, 1]$ and $\pi(I) := \bigcup_{n=1}^N \pi(I_n)$ for $I := \bigcup_{n=1}^N I_n$, where I_n are disjoint intervals. The Lebesgue measure on $[0, 1]$ is denoted as μ. Then, we have $\mu(I) = \mathbb{Q}[\pi(I)]$ for any interval $I \subseteq [0, 1]$.

4.1 Super-Replication

Consider the following self-financing strategy $\mathcal{G}^B(K; B_1, B_2)$ for $\forall K \in [0, B_1)$:

1. At the initial outset
   - Buy $\frac{1}{B_1 - K}$ units of a call option with strike K.
   - Sell $\frac{1}{B_1 - K}$ units of a call option with strike B_2.
   - Buy $\frac{B_2 - B_1}{B_1 - K}$ units of the one-touch option with a barrier level B_2.
   - Sell $\frac{B_2 - K}{B_1 - K}$ units of a digital call option with strike B_2.

2. At the first time of hitting B_1
   - Sell $\frac{1}{B_1-K}$ units of the forward contract
3. At the first time of hitting B_2
   - Buy $\frac{1}{B_1-K}$ units of the forward contract.

Fig.1 shows that the $\mathcal{G}^B(K;B_1,B_2)$ strategy super-replicates the touch/no-touch option with $K \in [0, B_1)$. We investigate the optimal strategies properties. First, we define the following.

Definition 3. *The initial value of the $\mathcal{G}^B(K;B_1,B_2)$ strategy is defined as*

$$G^B(K;B_1,B_2) := \frac{C(K)-C(B_2)}{B_1-K} + \frac{B_2-B_1}{B_1-K}O(B_2) + \frac{B_2-K}{B_1-K}\partial_K^- C(B_2), \tag{4.1}$$

$G_*^B(B_1,B_2)$ as the infimum value of $G^B(K;B_1,B_2)$ with respect to K, $K_G^B(B_1,B_2)$ as a strike price by which the infimum is attained:

$$\begin{aligned} G_*^B(B_1,B_2) &:= \inf_{K\in(-\infty,B_1)} G^B(K;B_1,B_2) \\ &= G^B(K_G^B(B_1,B_2);B_1,B_2) \end{aligned} \tag{4.2}$$

and $\mathcal{G}_*^B(B_1,B_2)$ as the corresponding strategy.

There is another super-replication: the $\mathcal{G}_*(B_1)$ strategy combined with a short position of the one-touch option with barrier B_2. The following theorem states that the better of the two strategies is the sharp upper bound, because the bound is attained by an expectation of the payoff with respect to a certain martingale.

Theorem 1. *If the set $\mathcal{N}(\partial_K G^B(B_1,B_2))$ is not empty, the infimum of Eq.(4.2) is attained by any element of a set $\mathcal{N}(\partial_K G^B(B_1,B_2))$, an interval of $(-\infty,B_1)$. For all $K_G^B \in \mathcal{N}(\partial_K G^B(B_1,B_2))$, the following holds:*

$$\begin{aligned} & \mathbb{Q}\left[K_+ < S_T < B_2\right] \\ \leq\ & G_*^B(B_1,B_2) = \mathbb{E}\left[\frac{S_T-K_G^B}{B_1-K_G^B}\ :\ K_G^B < S_T < B_2\right] + \frac{B_2-B_1}{B_1-K_G^B}\mathbb{Q}\left[\tau_2 \leq T\right] \\ \leq\ & \mathbb{Q}\left[K_- \leq S_T < B_2\right], \end{aligned} \tag{4.3}$$

where $K_- := \inf\mathcal{N}(\partial_K G^B(B_1,B_2))$ and $K_+ := \sup\mathcal{N}(\partial_K G^B(B_1,B_2))$. If the set $\mathcal{N}(\partial_K G^B(B_1,B_2))$ is empty, the infimum of Eq.(4.2) is not attained and $G_*^B(B_1,B_2) = \mathbb{Q}\left[S_T < B_2\right]$.

If $G_*^B(B_1,B_2) < G_*(B_1) - O(B_2)$, then $\mathcal{N}(\partial_K G^B(B_1,B_2))$ is a non-empty interval of $(\sup\mathcal{N}(\partial_K G(B_1)), B_1)$.

In addition, there is a continuous martingale process $\left\{S_t^G\right\}_{t\in[0,T]}$ such that

$$\min\left\{G_*^B(B_1,B_2), G_*(B_1) - O(B_2)\right\} = \mathbb{Q}\left[\tau_1\left(S^G\right) \leq T < \tau_2\left(S^G\right)\right]. \tag{4.4}$$

Proof. First, by differentiating G^B with respect to K, we obtain

$$\begin{aligned} \partial_K^- G^B(K) &= \frac{1}{B_1-K}\partial_K^- C(K) + \frac{C(K)-C(B_2)}{(B_1-K)^2} + \frac{B_2-B_1}{(B_1-K)^2}\left(O(B_2) + \partial_K^- C(B_2)\right) \\ &= \frac{1}{B_1-K}\left(G^B(K) + \partial_K^- C(K) - \partial_K^- C(B_2)\right) \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} \partial_{KK}^- G^B(K) &= \frac{1}{B_1-K}\partial_{KK}^- C(K) + \frac{2}{(B_1-K)^2}\partial_K^- C(K) + 2\frac{C(K)-C(B_2)}{(B_1-K)^3} \\ &\quad +2\frac{B_2-B_1}{(B_1-K)^3}\left(O(B_2) + \partial_K^- C(B_2)\right) \\ &= \frac{1}{B_1-K}\partial_{KK}^- C(K) + \frac{2}{B_1-K}\partial_K^- G^B(K). \end{aligned} \tag{4.6}$$

Note that $\partial_K^- G^B$ takes at least one positive value, because

$$\lim_{K\to B_1}(B_1-K)^2\partial_K^- G^B(K) = C(B_1) - C(B_2) + (B_2 - B_1)\left(O(B_2) + \partial_K^- C(B_2)\right) > 0. \tag{4.7}$$

Since $\partial_{KK}^- G^B \geq 0$ if $\partial_K^- G^B \geq 0$, $\mathcal{N}(\partial_K G^B(B_1, B_2))$ is empty or a non-empty interval. If $\mathcal{N}(\partial_K G^B(B_1, B_2))$ is empty, we have

$$G^*(B_1, B_2) = \lim_{K\to-\infty} G(K; B_1, B_2) = \mathbb{Q}\left[S_T < B_2\right]. \tag{4.8}$$

If $\mathcal{N}(\partial_K G^B(B_1, B_2))$ is not empty, we have Eq.(4.3). Moreover, if the following holds:

$$G_*^B(B_1, B_2) < G_*(B_1) - O(B_2), \tag{4.9}$$

then $\mathcal{N}(\partial_K G^B(B_1, B_2))$ is a non-empty interval of $(\sup \mathcal{N}(\partial_K G(B_1)), B_1)$, because

$$\mathbb{Q}\left[K_G^B < S_T < B_2\right] \leq G_*^B < \mathbb{Q}\left[K_G(B_1) \leq S_T < B_2\right] - \left(O(B_2) - \mathbb{Q}\left[B_2 \leq S_T\right]\right), \tag{4.10}$$

where any $K_G^B \in \mathcal{N}(\partial_K G^B(B_1, B_2))$ and $K_G(B_1) \in \mathcal{N}(\partial_K G(B_1))$. Note that if $\mathcal{N}(\partial_K G^B(B_1, B_2))$ is not empty, we have

$$\begin{aligned}
\mathbb{E}\left[S_T - B_1 : \pi\left(\left[k_G^B, b_2\right)\right)\right] &= \mathbb{E}\left[S_T - K_G^B : \pi\left(\left[k_G^B, b_2\right)\right)\right] + (K_G^B - B_1)\mu\left(\left[k_G^B, b_2\right)\right) \\
&= C(K_G^B) - C(B_2) - (B_2 - K_G^B)\mu\left([b_2, 1]\right) + (K_G^B - B_1)\mu\left(\left[k_G^B, b_2\right)\right) \\
&= C(K_G^B) - C(B_2) - (B_2 - B_1)\mu\left([b_2, 1]\right) - (B_1 - K_G^B)\mu\left(\left[k_G^B, 1\right]\right) \\
&= (B_1 - B_2)\mathbb{Q}\left[\tau_2 \leq T\right],
\end{aligned} \tag{4.11}$$

where $b_2 = \mathbb{Q}[S_T < B_2]$ and $k_G^B = b_2 - G_*^B$, and if $\mathcal{N}(\partial_K G^B(B_1, B_2))$ is empty, we have

$$\begin{aligned}
\mathbb{E}\left[S_T - B_1 : \pi\left(\left[k_G^B, b_2\right)\right)\right] &= \lim_{K\to-\infty}(B_1 - K)^2\partial_K^- G^B(K) + (B_1 - B_2)\mathbb{Q}\left[\tau_2 \leq T\right] \\
&= (B_1 - B_2)\mathbb{Q}\left[\tau_2 \leq T\right].
\end{aligned} \tag{4.12}$$

Next, suppose that Eq.(4.9) holds. We show that there an interval $[x, y] \subset \left[0, k_G^B\right]$ exists such that

$$\mu\left([x, y) \cup [b_2, 1]\right) = \mathbb{Q}\left[\tau_2 \leq T\right] \tag{4.13}$$

and

$$\mathbb{E}\left[S_T - B_2 : \pi([x, y) \cup [b_2, 1])\right] = 0. \tag{4.14}$$

Let $x = 0$ and y be a real number satisfied with Eq.(4.13) with $x = 0$. Then, since $y \geq k_L^{(2)} := L_*(B_2) - (1 - b_2)$, we have

$$\begin{aligned}
\mathbb{E}\left[S_T - B_2 : \pi([0, y) \cup [b_2, 1])\right] &\leq \mathbb{E}\left[S_T - B_2 : \pi([0, k_L^{(2)}) \cup [b_2, 1])\right] \\
&= 0.
\end{aligned} \tag{4.15}$$

Conversely, let $y = k_G^B$ and x be a real number satisfied with Eq.(4.13) with $y = k_G^B$. By Eq.(4.9), we have

$$\begin{aligned}
\mu\left(\left[k_G^B, b_2\right)\right) &< G_*(B_1) - \mu\left(\left[x, k_G^B\right) \cup [b_2, 1]\right) \\
&= \mu\left(\left[k_G^{(1)}, 1\right)\right) - \mu\left(\left[x, k_G^B\right) \cup [b_2, 1]\right),
\end{aligned} \tag{4.16}$$

where $k_G^{(1)} := 1 - G_*(B_1)$. Then, we have $x > k_G^{(1)}$. In addition, by Eq.(4.11), we have

$$\begin{aligned}
\mathbb{E}\left[S_T - B_2 \,:\, \pi\left(\left[x, k_G^B\right) \cup [b_2, 1]\right)\right] &= \mathbb{E}\left[S_T - B_1 \,:\, \pi\left(\left[x, k_G^B\right) \cup [b_2, 1]\right)\right] + (B_1 - B_2)\mathbb{Q}\left[\tau_2 \le T\right] \\
&\ge \mathbb{E}\left[S_T - B_1 \,:\, \pi\left(\left[k_G^{(1)}, k_G^B\right) \cup [b_2, 1]\right)\right] + (B_1 - B_2)\mathbb{Q}\left[\tau_2 \le T\right] \\
&= -\mathbb{E}\left[S_T - B_1 : \pi\left(\left[k_G^B, b_2\right)\right)\right] + (B_1 - B_2)\mathbb{Q}\left[\tau_2 \le T\right] \\
&= 0. \qquad (4.17)
\end{aligned}$$

Therefore, we can find an interval $[x, y)$ and have

$$\begin{aligned}
\mathbb{E}\left[S_T : \pi\left([x, y) \cup \left[k_G^B, 1\right]\right)\right] &= B_2\mu\left([x, y) \cup [b_2, 1]\right) + \mathbb{E}\left[S_T : \pi\left(\left[k_G^B, b_2\right)\right)\right] \\
&= B_1\mu\left([x, y) \cup \left[k_G^B, 1\right)\right), \qquad (4.18)
\end{aligned}$$

using Eq.(4.11) again. Then, we construct a martingale $\left\{S_t^G\right\}_{t\in[0,T]}$. Let X_1 and X_2 be random variables defined as

$$X_1 := \begin{cases} B_1, & \pi\left([x, y) \cup \left[k_G^B, 1\right]\right) \\ \beta_1, & \textit{otherwise} \end{cases}, \qquad (4.19)$$

and

$$X_2 := \begin{cases} B_2, & \pi\left([x, y) \cup [b_2, 1]\right) \\ \beta_2, & \pi\left(\left[k_G^B, b_2\right)\right) \\ \beta_1, & \textit{otherwise} \end{cases}, \qquad (4.20)$$

where $\beta_1 \in [0, B_1)$, $\beta_2 \in [0, B_2)$ are taken as in Lemma 1 and S_t^* is a stochastic process defined as

$$S_t^* := S_0 1_{\{t < \frac{1}{3}T\}} + X_1 1_{\{\frac{1}{3}T \le t < \frac{2}{3}T\}} + X_2 1_{\{\frac{2}{3}T \le t < T\}} + S_T 1_{\{t = T\}}. \qquad (4.21)$$

Then, applying the same argument from Lemma 1 to $\{S_t^*\}_{t\in[0,T]}$, we obtain a continuous martingale with respect to a certain filtration. We obtain $\mathbb{Q}\left[\tau_1 \le T < \tau_2\right] = G_*^B$.

Finally, suppose that Eq.(4.9) does not hold. If $O(B_2) = G_*(B_2)$, we have $\mathbb{E}\left[S_T - B_2 \,:\, \pi\left(\left[k_G^{(2)}, 1\right]\right)\right] = 0$, where $k_G^{(2)} := 1 - G_*(B_2)$. If Eq.(4.9) holds with equality, we have

$$\begin{aligned}
\mathbb{E}\left[S_T - B_2 \,:\, \pi\left(\left[k_G^{(1)}, k_G^B\right] \cup [b_2, 1]\right)\right] &= \mathbb{E}\left[S_T - B_1 \,:\, \pi\left(\left[k_G^{(1)}, k_G^B\right] \cup [b_2, 1]\right)\right] + (B_1 - B_2)\mu\left(\left[k_G^{(1)}, k_G^B\right] \cup [b_2, 1]\right) \\
&= -\mathbb{E}\left[S_T - B_1 \,:\, \pi\left(\left[k_G^B, b_2\right]\right)\right] + (B_1 - B_2)\mu\left(\left[k_G^{(1)}, k_G^B\right] \cup [b_2, 1]\right) \\
&= 0. \qquad (4.22)
\end{aligned}$$

Then, we can take an interval $[x, y) \subseteq \left[k_G^{(1)}, b_2\right)$ which is satisfied with Eq.(4.13) and Eq.(4.14), because of $k_G^{(1)} < k_G^{(2)}$. Similar to the previous case, a continuous martingale can be constructed such that Eq.(4.4) holds. □

4.2 Sub-Replication

Consider the following self-financing strategy $\mathcal{L}^B(K; B_1, B_2)$ for $\forall K \in [0, B_1)$:

1. At the initial outset
   - Sell $\frac{1}{B_1 - K}$ units of a put option with strike K.

- Buy $\frac{B_2-B_1}{B_1-K}$ units of the one-touch option with a barrier level B_2.

2. At the first time of hitting B_1
   - Sell $\frac{1}{B_1-K}$ units of a forward contract.
3. At the first time of hitting B_2
   - Buy $\frac{1}{B_1-K}$ units of the forward contract.

Fig.2 shows that the $\mathcal{L}^B(K;B_1,B_2)$ strategy sub-replicates the touch/no-touch option with $K\in[0,B_1)$. We investigate the optimal strategy properties. First, we define the following.

Definition 4. *The initial value of the strategy $\mathcal{L}^B(K;B_1,B_2)$ is defined as*

$$L^B(K;B_1,B_2) := \frac{-P(K)}{B_1-K} + \frac{B_2-B_1}{B_1-K}O(B_2), \tag{4.23}$$

$L^B_*(B_1,B_2)$ as the supremum value of $L^B(K;B_1,B_2)$ with respect to K, $K^B_L(B_1,B_2)$ as a strike price by which the supremum is attained:

$$\begin{aligned} L^B_*(B_1,B_2) &:= \sup_{K\in(-\infty,B_1)} L^B(K;B_1,B_2) \\ &= L^B(K^B_L(B_1,B_2);B_1,B_2) \end{aligned} \tag{4.24}$$

and $\mathcal{L}^B_*(B_1,B_2)$ as the corresponding strategy.

There is another sub-replication: the strategy $\mathcal{L}_*(B_1)$ combined with a short position of the one-touch option with barrier B_2. The following theorem states that the better of the two strategies is the sharp lower bound, because the bound is attained by an expectation of the payoff with respect to a certain martingale.

Theorem 2. *The supremum of Eq.(4.24) is attained by any element of $\mathcal{N}(\partial_K L^B(B_1,B_2))$, an interval of $(0,\sup\mathcal{N}(\partial_K L(B_1))]$. For all $K^B_L\in\mathcal{N}(\partial_K L^B(B_1,B_2))$, the following holds:*

$$\mathbb{Q}\left[S_T<K_-\right] \le L^B_*(B_1,B_2) = \mathbb{E}\left[\frac{S_T-K^B_L}{B_1-K^B_L} : S_T\le K^B_L\right] + \frac{B_2-B_1}{B_1-K^B_L}\mathbb{Q}\left[\tau_2\le T\right] \le \mathbb{Q}\left[S_T\le K_+\right], \tag{4.25}$$

where $K_- := \inf\mathcal{N}(\partial_K L^B(B_1,B_2))$ and $K_+ := \sup\mathcal{N}(\partial_K L^B(B_1,B_2))$.

In addition, there is a martingale process $\left\{S^L_t\right\}_{t\in[0,T]}$ such that

$$\max\left\{L^B_*(B_1,B_2), L_*(B_1)-O(B_2)\right\} = \mathbb{Q}\left[\tau_1\left(S^L\right)\le T<\tau_2\left(S^L\right)\right]. \tag{4.26}$$

Proof. First, by differentiating L^B with respect to K, we obtain

$$\begin{aligned} \partial^-_K L^B(K) &= \frac{-1}{B_1-K}\partial^-_K P(K) - \frac{P(K)}{(B_1-K)^2} + \frac{B_2-B_1}{(B_1-K)^2}O(B_2) \\ &= \frac{1}{B_1-K}\left(-\partial^-_K P(K) + L^B(K)\right) \end{aligned} \tag{4.27}$$

and

$$\begin{aligned} \partial^-_{KK} L^B(K) &= \frac{-1}{B_1-K}\partial^-_{KK} P(K) - \frac{2}{(B_1-K)^2}\partial^-_K P(K) - 2\frac{P(K)}{(B_1-K)^3} + 2\frac{B_2-B_1}{(B_1-K)^3}O(B_2) \\ &= \frac{2}{B_1-K}\partial^-_K L^B(K) - \frac{1}{B_1-K}\partial^-_{KK} P(K). \end{aligned} \tag{4.28}$$

Note that $\partial_K^- L^B(0) = \frac{B_2-B_1}{B_1^2}O(B_2) > 0$ and by Eq.(3.11)

$$\begin{aligned}\partial_K^+ L^B(K_+) &= \frac{-1}{B_1-K_+}\partial_K^+ P(K_+) - \frac{P(K_+)}{(B_1-K_+)^2} + \frac{B_2-B_1}{(B_1-K_+)^2}O(B_2) \\ &\leq -\frac{C(B_1)}{(B_1-K_+)^2} + \frac{B_2-B_1}{(B_1-K_+)^2}O(B_2) \\ &\leq 0, \qquad (4.29)\end{aligned}$$

where $K_+ := \sup \mathcal{N}(\partial_K L(B_1))$ and ∂_K^+ is the right-sided derivative operator. Since $\partial_{KK}^- L^B \leq 0$ if $\partial_K^- L^B \leq 0$, $\mathcal{N}(\partial_K L^B(B_1, B_2))$ is an interval of $(0, K_+]$ and we have Eq.(4.25). Note that

$$\begin{aligned}\mathbb{E}\left[B_1 - S_T : \pi\left(\left[0, k_L^B\right)\right)\right] &= \mathbb{E}\left[K_L^B - S_T : \pi\left(\left[0, k_L^B\right)\right)\right] + \mathbb{E}\left[B_1 - K_L^B : \pi\left(\left[0, k_L^B\right)\right)\right] \\ &= P(K_L^B) + (B_1 - K_L^B)\mu\left(\left[0, k_L^B\right)\right) \\ &= (B_2 - B_1)\mathbb{Q}\left[\tau_2 \leq T\right], \qquad (4.30)\end{aligned}$$

where $k_L^B := L_*^B$.

Next, suppose that the following holds:

$$L_*(B_1) - O(B_2) < L_*^B(B_1, B_2). \qquad (4.31)$$

We show that there exists an interval $[x, y] \subset \left[k_L^B, b_1\right]$, where $b_1 = \mathbb{Q}[S_T < B_1]$, such that

$$\mu\left([x, y) \cup [b_1, 1]\right) = \mathbb{Q}\left[\tau_2 \leq T\right] \qquad (4.32)$$

and

$$\mathbb{E}\left[S_T - B_2 : \pi\left([x, y) \cup [b_1, 1]\right)\right] = 0. \qquad (4.33)$$

We can take an interval that satisfies Eq.(4.32) because Eq.(4.31) implies

$$\begin{aligned}\mathbb{Q}\left[\tau_2 \leq T\right] &> \mu\left(\left[0, k_L^{(1)}\right) \cup [b_1, 1]\right) - \mu\left(\left[0, k_L^B\right)\right) \\ &\geq \mu\left([b_1, 1]\right), \qquad (4.34)\end{aligned}$$

where $k_L^{(1)} := L^*(B_1) - (1 - b_1)$. Let $y = b_1$ and x be a solution of Eq.(4.32) with $y = b_1$. We have $x \geq k_G^{(2)} := 1 - G_*(B_2)$ because $\mathbb{Q}\left[\tau_2 \leq T\right] \leq \mu\left([k_G^{(2)}, 1]\right)$. Then, we have

$$\begin{aligned}\mathbb{E}\left[S_T - B_2 : \pi([x, 1))\right] &\geq \mathbb{E}\left[S_T - B_2 : \pi\left(\left[k_G^{(2)}, 1\right)\right)\right] \\ &= 0. \qquad (4.35)\end{aligned}$$

Conversely, let $x = k_L^B$ and y be the solution of Eq.(4.32) with $x = k_L^B$. We have $y \geq k_L^{(1)}$, because $\mathbb{Q}\left[\tau_2 \leq T\right] > \mu\left(\left[k_L^B, k_L^{(1)}\right) \cup [b_1, 1]\right)$ by Eq.(4.31). Then, by Eq.(4.30), we have

$$\begin{aligned}\mathbb{E}\left[S_T - B_2 : \pi\left(\left[k_L^B, y\right) \cup [b_1, 1]\right)\right] &= \mathbb{E}\left[S_T - B_1 : \pi\left(\left[k_L^B, y\right) \cup [b_1, 1]\right)\right] + (B_1 - B_2)\mathbb{Q}\left[\tau_2 \leq T\right] \\ &\leq \mathbb{E}\left[S_T - B_1 : \pi\left(\left[k_L^B, k_L^{(1)}\right) \cup [b_1, 1]\right)\right] + (B_1 - B_2)\mathbb{Q}\left[\tau_2 \leq T\right] \\ &= \mathbb{E}\left[B_1 - S_T : \pi\left(\left[0, k_L^B\right)\right)\right] + (B_1 - B_2)\mathbb{Q}\left[\tau_2 \leq T\right] \\ &= 0. \qquad (4.36)\end{aligned}$$

Therefore, we can find the interval $[x, y)$ and we have for this interval

$$\begin{aligned}\mathbb{E}\left[S_T : \pi\left(\left[0, k_L^B\right) \cup [x, y) \cup [b_1, 1]\right)\right] &= \mathbb{E}\left[S_T : \pi\left(\left[0, k_L^B\right)\right)\right] + B_2\mu([x, y) \cup [b_1, 1]) \\ &= B_1\mu\left(\left[0, k_L^B\right) \cup [x, y) \cup [b_1, 1]\right), \end{aligned} \tag{4.37}$$

using Eq.(4.30) again. Then, we construct a martingale $\left\{S_t^L\right\}_{t\in[0,T]}$. Let X_1 and X_2 be random variables defined as

$$X_1 := \begin{cases} B_1, & \pi\left(\left[0, k_L^B\right) \cup [x, y) \cup [b_1, 1]\right) \\ \beta_1, & otherwise \end{cases}, \tag{4.38}$$

and

$$X_2 := \begin{cases} B_2, & \pi([x, y) \cup [b_1, 1]) \\ \beta_2, & \pi\left(\left[0, k_L^B\right)\right) \\ \beta_1, & otherwise \end{cases}, \tag{4.39}$$

where $\beta_1 \in [0, B_1)$, $\beta_2 \in [0, B_2)$ are taken as in Lemma 1 and S_t^* be a stochastic process defined as

$$S_t^* := S_0 1_{\{t<\frac{1}{3}T\}} + X_1 1_{\{\frac{1}{3}T\le t<\frac{2}{3}T\}} + X_2 1_{\{\frac{2}{3}T\le t<T\}} + S_T 1_{\{t=T\}}. \tag{4.40}$$

Then, applying the same argument from Lemma 1 to $\{S_t^*\}_{t\in[0,T]}$, we obtain a continuous martingale with respect to a certain filtration. We obtain $\mathbb{Q}[\tau_1 \le T < \tau_2] = L_*^B$.

Finally, suppose that Eq.(4.31) does not hold. If $O(B_2) = L_*(B_2)$ and let $k_L^{(2)} := L_*(B_2) - (1 - b_2)$, we have $\mathbb{E}\left[S_T - B_2 \ : \ \pi([0, k_L^{(2)}) \cup [b_2, 1])\right] = 0$. If Eq.(4.31) holds with equality, we have by Eq.(4.30)

$$\begin{aligned}\mathbb{E}\left[S_T - B_2 \ : \ \pi\left(\left[k_L^B, k_L^{(1)}\right) \cup [b_1, 1]\right)\right] &= \mathbb{E}\left[S_T - B_1 \ : \ \pi\left(\left[k_L^B, k_L^{(1)}\right) \cup [b_1, 1]\right)\right] + (B_1 - B_2)\mathbb{Q}[\tau_2 \le T] \\ &= -\mathbb{E}\left[S_T - B_1 \ : \ \pi\left(\left[0, k_L^B\right)\right)\right] + (B_1 - B_2)\mathbb{Q}[\tau_2 \le T] \\ &= 0. \end{aligned} \tag{4.41}$$

Note that $k_L^{(2)} < k_L^{(1)}$, because $(B - K)^2\partial_K^- L(K; B)$ is decreasing with respect to B and K. Then, we can take a set $D \subseteq [0, k_L^{(1)}] \cup [b_1, b_2)$ which is satisfied with

$$\mu\left(D \cup [b_2, 1]\right) = \mathbb{Q}[\tau_2 \le T] \tag{4.42}$$

and

$$\mathbb{E}\left[S_T - B_2 : \pi\left(D \cup [b_2, 1]\right)\right] = 0. \tag{4.43}$$

Similar to the previous case, a continuous martingale can be constructed such that Eq.(4.26) holds. □

4.3 The Finite Basis Situation

In this section, we consider the case where only a finite number of strikes are given. Suppose that call options with strikes $K_0 < K_1 < \cdots < K_N$, where $K_0 = 0$ and $B_2 \le K_N$, are traded with no-arbitrage prices $\{C_n\}_{n=0}^N$. We consider super-replication and sub-replication for the touch/no-touch option with barrier levels B_1 and B_2 using the one-touch option with a barrier level B_2. We assume that a no-arbitrage price of the digital call option with strike B_2 is given as D_2 in case of the super-replication and that with strike B_1

is given as D_1 in case of the sub-replication. Here, a no-arbitrage price D of digital call option with strike $B \in (K_{n-1}, K_n]$ satisfies

$$-\frac{C_{n+1} - C_n}{K_{n+1} - K_n} \leq D \leq -\frac{C_{n-1} - C(B)}{K_{n-1} - B}, \tag{4.44}$$

where $C(B) := \frac{C_{n+1}-C_n}{K_{n+1}-K_n}(B - K_n) + C_n$. Even if these digital call options are not liquid, we can regard the lower bound as the digital call price with strike B_2 in case of the super-replication and the upper bound as the digital call price with strike B_1 in case of the sub-replication.

First, we consider the super-replication. We suppose that the no-touch option with a barrier level B_2 is traded and the price of this no-touch option satisfies

$$\sup_{K_n < B_2} L(K_n; B_2) \leq O(B_2) \leq \inf_{K_n < B_2} G(K_n; B_2). \tag{4.45}$$

The upper bound on the touch/no-touch option derived from the super-replication is

$$\min \left\{ G_*^B \left(B_1, B_2; \{K_n\}_{n=0}^N \right), G_* \left(B_1; \{K_n\}_{n=0}^N \right) - O(B_2) \right\}, \tag{4.46}$$

where $G_*^B \left(B_1, B_2; \{K_n\}_{n=0}^N\right) := \inf_{K_n < B_1} G^B(K_n; B_1, B_2)$ and $G_* \left(B_1; \{K_n\}_{n=0}^N\right) := \inf_{K_n < B_1} G(K_n; B_1)$. Although the marginal distribution of S_T is not uniquely specified in this case, the following corollary shows that there is a distribution consistent with the given option prices under which we can construct a martingale attaining the upper bound.

Corollary 1. *There is a distribution μ_C of S_T which is consistent with the given call prices, the given digital call option with a strike B_2 and the given no-touch option with a barrier level B_2 satisfying Eq.(4.45) such that Eq.(4.46) is equal to Eq.(4.4) with distribution μ_C.*

Proof. First, we assume $B_2 \in \{K_n\}_{n=0,\cdots,N}$ and $D_2 = -\frac{C_{n-1}-C_n}{K_{n-1}-K_n}$, where $B_2 = K_n$, Let us consider call options prices $\{C(K)\}_{K \in [0,+\infty)}$: $C(K)$ is the linear interpolation of C_n if $K \in [K_0, K_N]$ and an arbitrary extrapolation excluding arbitrage opportunities if $K \in [K_N, +\infty)$. Let μ_C be a distribution implied by the call option prices C. We can apply Proposition 1 and 2 with the distribution μ_C to the no-touch option with a barrier level B_2 and obtain the optimal strikes $K_G(B_2)$ and $K_L(B_2)$. These may not be uniquely determined, but can be taken as one of the given strikes, since the distribution μ_C consists of atoms at K_n on $[0, B_2)$. Hence, the distribution μ_C is consistent with Eq.(4.45). By the same reason, Eq.(4.4) with distribution μ_C is attained by one of the given strikes. Then, Eq.(4.46) is equal to Eq.(4.4) with distribution μ_C.

In the general case, two call prices, $C(\tilde{K})$ and $C(B_2)$, with strikes, $\tilde{K} := B_2 - \varepsilon$ and B_2, can be added into the given call price set as: $C(K) := -D_2(K - \hat{K}) + \hat{C}$ for $K = \tilde{K}, B_2$, where $B_2 \in (K_{n-1}, K_n]$, $(\hat{K}, \hat{C}) = (K_{n-1}, C_{n-1})$ in case of $D_2 > -\frac{C_{n-1}-C_n}{K_{n-1}-K_n}$, $(\hat{K}, \hat{C}) = (K_n, C_n)$ in the other case, ε is a sufficiently small positive value such that $\tilde{K}$ is not the optimal strike for $G_*(B_2)$ and $L_*(B_2)$. Then, the same argument from the first case can be applied. □

Next, we consider the sub-replication. This is more involved than the super-replication. The lower bound on the touch/no-touch option derived from the sub-replication is

$$\max \left\{ L_*^B \left(B_1, B_2; \{K_n\}_{n=0}^N \right), L_* \left(B_1; \{K_n\}_{n=0}^N \right) - O(B_2) \right\}, \tag{4.47}$$

where $L_*^B \left(B_1, B_2; \{K_n\}_{n=0}^N\right) := \sup_{K_n < B_1} L^B(K_n; B_1, B_2)$ and $L_* \left(B_1; \{K_n\}_{n=0}^N\right) := \sup_{K_n < B_1} L(K_n; B_1)$. We assume $B_1 \in \{K_n\}_{n=0,\cdots,N}$ and

$$\sup_{K_n < B_2} L(K_n; B_2) < O(B_2) < \inf_{K_n < B_2} G(K_n; B_2). \tag{4.48}$$

Owing to these assumptions, we have the similar result to Corollary 1.

Corollary 2. *There is a distribution μ_C of S_T which is consistent with the given call prices which includes that with strike B_1, the given digital call option with a strike B_1 and the given no-touch option with a barrier level B_2 satisfying Eq.(4.48) such that Eq.(4.47) is equal to Eq.(4.26) with distribution μ_C.*

Proof. Let $n \in \{0, 1, \cdots, N\}$ be such that $B_1 = K_n$. The proof is the same as the first part of Corollary 1, if $D_1 = -\frac{C_{n-1}-C_n}{K_{n-1}-K_n}$. In the general case, let $\tilde{K} := B_1 - \varepsilon > K_{n-1}$ for a sufficiently small positive value ε and $C(\tilde{K}) := -D_1(\tilde{K} - K_n) + C_n$. We can take ε such that

$$\sup_{\tilde{K}, K_i < B_2} L(K_i; B_2) < O(B_2) < \inf_{\tilde{K}, K_i < B_2} G(K_i; B_2). \tag{4.49}$$

Let μ_C be a distribution implied by an interpolation of the given call option prices C and $C(\tilde{K})$. Then, we have the conclusion for the distribution μ_C by the same argument from Corollary 1. □

5 Numerical Examples

This section provides numerical examples.

We regard Heston's stochastic volatility model (Heston (1993)) as the underlying asset process. The process underlying the Heston model is as follows:

$$\frac{dS_t}{S_t} = (r - q)dt + \sigma_t dW_t, \tag{5.1}$$

$$d\sigma_t^2 = \kappa(\eta - \sigma_t^2)dt + \theta\sigma_t d\tilde{W}_t, \tag{5.2}$$

where W and $\tilde{W}$ are Brownian motions with correlation ρ under a risk-neutral measure. In addition, we assume that the parameters of the Heston model are as shown in Table 1.

r	q	σ_0^2	κ	η	θ	ρ
0.0	0.0	0.15^2	3.0	0.2^2	0.4	0.0

Table 1: Parameters of the Heston Model

The one-touch option considered has a 3-month maturity and a barrier level of 1.05 USD. We calculate the pricing bounds of our method, those of Brown et al. (2001) and exact prices by a Monte Carlo simulation with the initial spot price varied from 0.9 USD to 1.04 USD. We calculate pricing bounds derived from $\mathcal{G}_*^B$ and $\mathcal{L}_*^B$ strategies using another one-touch option with $B = 1.06$. This is evaluated by the Heston model with the same parameter set. The results are shown in Fig.3 and Table 2. Our lower bounds are proved to be higher than those of Brown et al. (2001) across the entire range and our upper bounds proved lower in the $[0.9, 0.98]$ range.

Additionally, Fig.4 shows a relationship between pricing bounds on the two one-touch options with barrier levels 1.05 USD and 1.06 USD, where the market conditions are the same as for the above example and the initial spot price is fixed at 1 USD. The pricing bounds of Brown et al. (2001) on the two one-touch options are $[0.315, 0.609]$ for the barrier level 1.05 USD and $[0.263, 0.529]$ for barrier level 1.06 USD. However, we established that a condition for no-arbitrage prices of these two options does not lie within these ranges but is within the range indicated in Fig.4.

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Figure 1: Payoff of Strategy $\mathcal{G}^B(K; B_1, B_2)$ with $S_0 = 1$, $K = 0.95$, $B_1 = 1.05$ and $B_2 = 1.06$

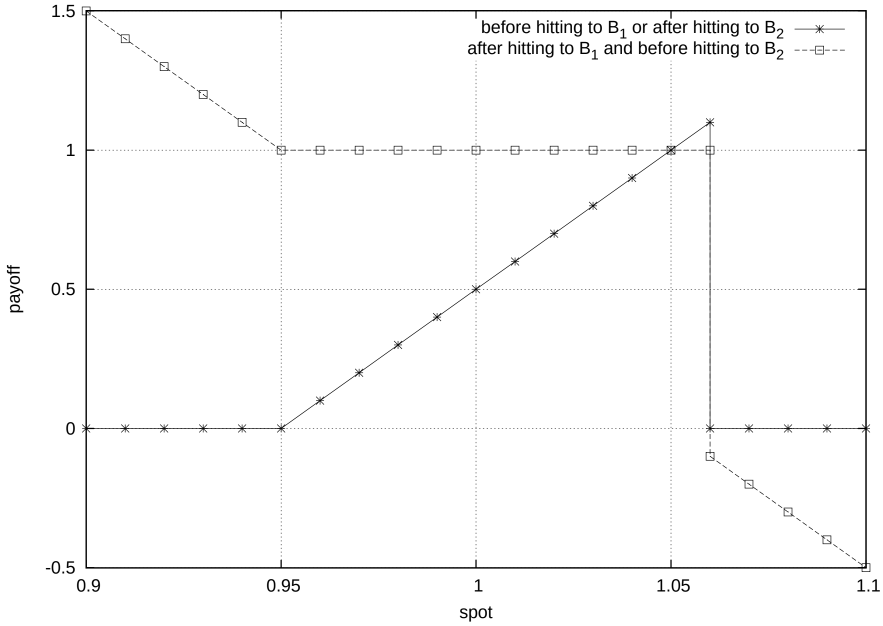

Figure 2: Payoff of Strategy $\mathcal{L}^B(K; B_1, B_2)$ with $S_0 = 1$, $K = 0.95$, $B_1 = 1.05$ and $B_2 = 1.06$

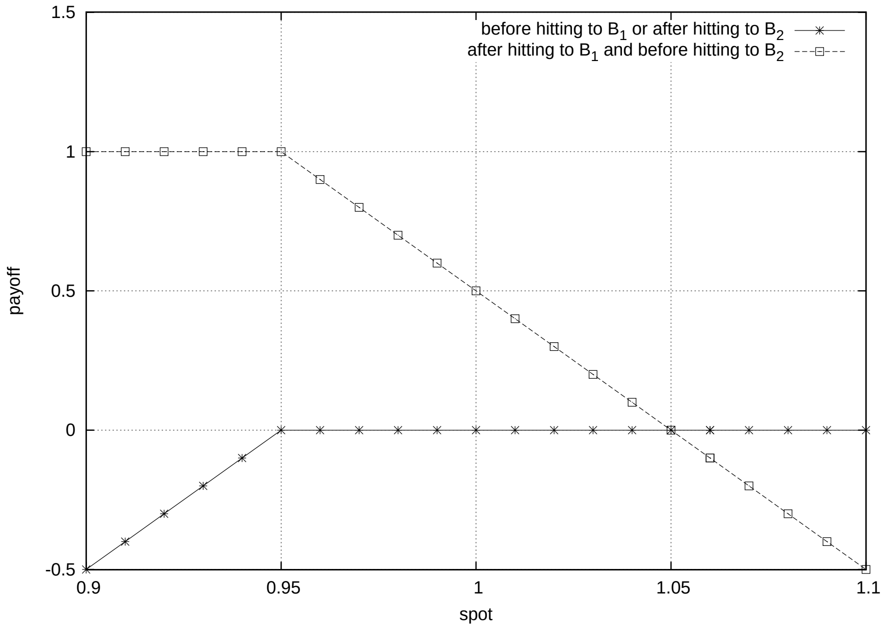

Figure 3: Pricing bounds on a one-touch option

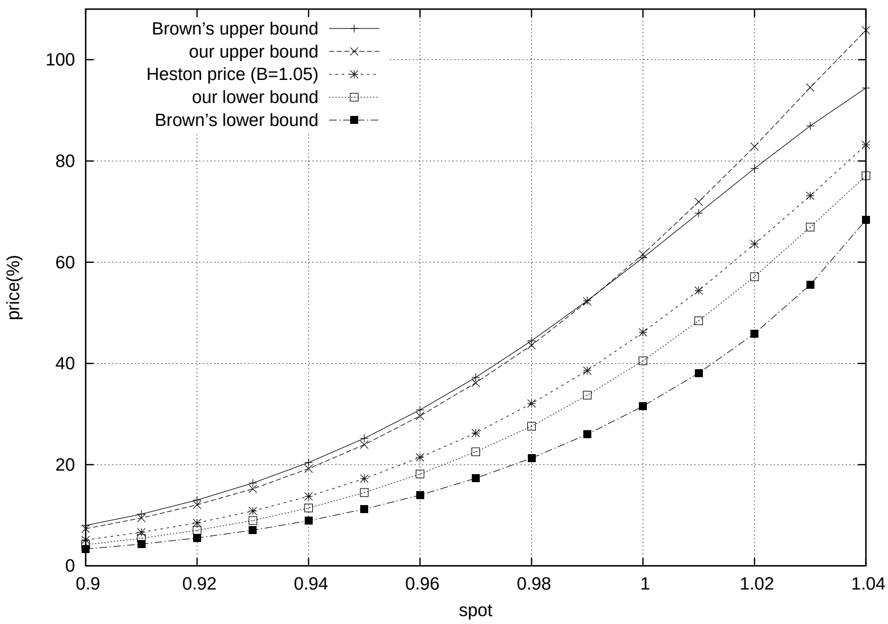

Table 2: Pricing bounds on a one-touch option (%)

spot	0.90	0.92	0.94	0.96	0.98	1.00	1.02	1.04
Brown et al. (2001)'s upper bound	8.0	13.0	20.4	30.8	44.5	60.9	78.5	94.4
Our upper bound W^G	7.3	12.1	19.2	29.6	43.6	61.6	82.8	105.8
Heston price ($B = 1.05$)	5.1	8.5	13.7	21.4	32.1	46.1	63.6	83.2
Our lower bound W^L	4.2	7.0	11.5	18.2	27.6	40.5	57.1	77.1
Brown et al. (2001)'s lower bound	3.3	5.5	8.9	14.0	21.3	31.5	45.9	68.3
Heston price ($B = 1.06$)	4.1	6.8	11.1	17.6	26.7	39.0	54.7	73.3

Figure 4: Pricing bounds on two one-touch options