Self-dual continuous processes

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Abstract

The important application of semi-static hedging in financial markets naturally leads to the notion of quasi self-dual processes which is, for continuous semimartingales, related to symmetry properties of both their ordinary as well as their stochastic logarithms. We provide a structure result for continuous quasi self-dual processes. Moreover, we give a characterisation of continuous Ocone martingales via a strong version of self-duality.

Keywords: self-duality, symmetric processes, Ocone martingales, semi-static hedging

1 Introduction

The duality principle in option pricing relates different financial products by a certain change of measure. It allows to transform complicated financial derivatives into simpler ones in a suitable dual market. For a comprehensive treatment, see [6, 7] and the literature cited therein.

Sometimes it is even possible to semi-statically hedge path-dependent barrier options with European ones. These are options which only depend on the asset price at maturity. Here semi-static refers to trading at most at inception and a finite number of stopping times like hitting times of barriers. The possibility of this hedge, however, requires a certain symmetry property of the asset price which has to remain

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invariant under the duality transformation, possibly after a power transform. This leads naturally to the concepts of self-duality, resp. quasi self-duality, see [2] and more recently [3, 15]. For references to the large literature of the special case of put-call symmetry, see [3, 8, 9, 10, 21].

Continuous symmetric processes have been characterised in [21], and it is shown therein that the conditional symmetry property is related to the self-duality of their stochastic exponentials. We extend this study by exploring the structure of quasi self-dual processes as well as characterising continuous Ocone martingales using results from [22] and a strong version of self-duality. Ocone martingales are a very important class of conditionally symmetric martingales; indeed, Tehranchi raised in [21] the question whether *all* conditionally symmetric martingales are Ocone. This question is still open. We do provide, however, an example of a non-Ocone martingale in continuous time which is process, but not conditionally symmetric.

2 Definitions and general properties

We work on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ where unless otherwise stated, the filtration satisfies the usual conditions with $\mathcal{F}_0$ being trivial up to P-null sets, and fix a finite but arbitrary time horizon $T > 0$. All stochastic processes are RCLL and defined on $[0, T]$ unless otherwise stated. We understand positive and negative in the strict sense.

Definition 1 *Let M be an adapted process. M is **conditionally symmetric** if for any stopping time $\tau \in [0, T]$ and any non-negative Borel function f*

$$E\left[f\left(M_T - M_\tau\right) \middle| \mathcal{F}_\tau \right] = E\left[f\left(M_\tau - M_T\right) \middle| \mathcal{F}_\tau \right]. \tag{1}$$

Here it is permissible that both sides of the equation are infinite. If M is an integrable conditionally symmetric process, then condition (1) implies that M is a martingale by choosing $f(x) = x$ $(= x^+ - x^-)$.

Definition 2 *Let S be a positive adapted process. S is **self-dual** if for any stopping time $\tau \in [0, T]$ and any non-negative Borel function f we have*

$$E\left[f\left(\frac{S_T}{S_\tau}\right) \middle| \mathcal{F}_\tau \right] = E\left[\frac{S_T}{S_\tau} f\left(\frac{S_\tau}{S_T}\right) \middle| \mathcal{F}_\tau \right]. \tag{2}$$

These definitions are new, and are motivated by the fact that in applications to semi-static hedging one typically considers hitting times of barriers which are stopping times. They differ from the ones used in [21] who uses bounded measurable f instead, and in particular deterministic times. However, all corresponding results in [21] applied in this paper can be adapted to our setting.

In the case when S is a martingale, we can define a probability measure Q, the so-called *dual measure*, via

$$\frac{dQ}{dP} = \frac{S_T}{S_0}. \tag{3}$$

Similarly, if $E\left[\sqrt{S_T}\right] < \infty$, or $E\left[S_T^w\right] < \infty$ for a $w \in [0,1]$, respectively, we define probability measures H, sometimes called 'half measure', respectively P^w, via

$$\frac{dH}{dP} = \frac{\sqrt{S_T}}{E\left[\sqrt{S_T}\right]}, \quad \frac{dP^w}{dP} = \frac{S_T^w}{E\left[S_T^w\right]}. \tag{4}$$

Note that the integrability of $S_T = S_0 \exp(X_T)$ under P implies the existence of the moment generating function of X_T under H for an open interval including the origin, i.e. X_T has all moments under H.

By Bayes' formula, the self-duality condition (2) can be expressed for a martingale S in terms of the dual measure Q defined in (3) as

$$E_P\left[f\left(\frac{S_T}{S_\tau}\right)\middle| \mathcal{F}_\tau\right] = E_Q\left[f\left(\frac{S_\tau}{S_T}\right)\middle| \mathcal{F}_\tau\right]. \tag{5}$$

Lemma 3 ***([21], Lemma 3.2.)*** *A positive continuous martingale S is self-dual if and only if*

$$E_P\left[\left(\frac{S_T}{S_\tau}\right)^p\middle| \mathcal{F}_\tau\right] = E_P\left[\left(\frac{S_T}{S_\tau}\right)^{1-p}\middle| \mathcal{F}_\tau\right] \tag{6}$$

for all complex $p = a + ib$ with $a \in [0,1]$ and all stopping times $\tau \in [0,T]$.

For the measure H (corresponding to $w = 1/2$) the following proposition has been stated in slightly different settings in [3, 15] and [21], and also for $w = 1$, i.e. for Q. Similar unconditional multivariate results are given in [16].

Proposition 4 *Let $S = \exp(X)$ be a martingale. Then S is self-dual if and only if for any stopping time $\tau \in [0,T]$ and any non-negative Borel function f*

$$E_{P^w}\left[f\left(X_T - X_\tau\right)\middle| \mathcal{F}_\tau\right] = E_{P^{1-w}}\left[f\left(X_\tau - X_T\right)\middle| \mathcal{F}_\tau\right] \tag{7}$$

holds for at least one (and then necessarily for all) $w \in [0,1]$.

For the half measure we immediately obtain the following special case.

Corollary 5 *Let $S = \exp(X)$ be a martingale. Then S is self-dual if and only if X is conditionally symmetric with respect to H.*

Proof of Proposition 4. As a consequence of the martingale property of S and Hölder's inequality we have that both $E_P\left[e^{w(X_T - X_\tau)}\middle|\,\mathcal{F}_\tau\right]$, for all $w \in [0,1]$, as well as $\left|E_P\left[e^{p(X_T - X_\tau)}\middle|\mathcal{F}_\tau\right]\right|$, for all complex $p = a + ib$ with $a \in [0,1]$, $\tau \in [0,T]$, are finite a.s.

Let S be self-dual, $w \in [0,1]$, so that (6) implies the following two equalities:

$$E_P\left[e^{w(X_T - X_\tau)}\middle|\,\mathcal{F}_\tau\right] = E_P\left[e^{(1-w)(X_T - X_\tau)}\middle|\,\mathcal{F}_\tau\right], \tag{8}$$

$$E_P\left[e^{(w+i\theta)(X_T - X_\tau)}\middle|\,\mathcal{F}_\tau\right] = E_P\left[e^{(1-w-i\theta)(X_T - X_\tau)}\middle|\,\mathcal{F}_\tau\right], \tag{9}$$

for $\theta \in \mathbb{R}$. By applying Bayes' formula we obtain

$$E_{P^w}\left[e^{i\theta(X_T - X_\tau)}\middle|\,\mathcal{F}_\tau\right] = \frac{E_P\left[e^{(w+i\theta)(X_T - X_\tau)}\middle|\,\mathcal{F}_\tau\right]}{E_P\left[e^{w(X_T - X_\tau)}\middle|\,\mathcal{F}_\tau\right]},$$

$$E_{P^{1-w}}\left[e^{i\theta(X_\tau - X_T)}\middle|\,\mathcal{F}_\tau\right] = \frac{E_P\left[e^{(1-w-i\theta)(X_T - X_\tau)}\middle|\,\mathcal{F}_\tau\right]}{E_P\left[e^{(1-w)(X_T - X_\tau)}\middle|\,\mathcal{F}_\tau\right]},$$

so that in view of (8, 9) the r.h.s. coincide and so do the l.h.s. Since the conditional characteristic functions $(X_T - X_\tau)$ under P^w coincide with the ones of $(X_\tau - X_T)$ under P^{1-w}, we end up with (7) for the claimed cases.

On the other hand, for an arbitrary $w \in [0,1]$, the P-martingale property of S, and by Bayes' formula we see that the l.h.s. (and hence the r.h.s.) of the following equations coincide:

$$E_{P^w}\left[e^{-w(X_T - X_\tau)}\middle|\,\mathcal{F}_\tau\right] = E_P\left[e^{w(X_T - X_\tau)}\middle|\,\mathcal{F}_\tau\right]^{-1}, \tag{10}$$

$$E_{P^{1-w}}\left[e^{-w(X_\tau - X_T)}\middle|\,\mathcal{F}_\tau\right] = \frac{E_P\left[e^{X_T - X_\tau}\middle|\,\mathcal{F}_\tau\right]}{E_P\left[e^{(1-w)(X_T - X_\tau)}\middle|\,\mathcal{F}_\tau\right]} = E_P\left[e^{(1-w)(X_T - X_\tau)}\middle|\,\mathcal{F}_\tau\right]^{-1}. \tag{11}$$

Furthermore, we have for all complex $p = a + ib$ with $a \in [0,1]$ that

$$E_{P^w}\left[e^{(p-w)(X_T - X_\tau)}\middle|\,\mathcal{F}_\tau\right] = E_{P^{1-w}}\left[e^{(w-p)(X_T - X_\tau)}\middle|\,\mathcal{F}_\tau\right].$$

Combining this equality with the fact that the r.h.s. of (10) and (11) coincide we obtain the equality of the l.h.s. of the following two equations

$$E_{P^w}\left[\left.\frac{e^{p(X_T-X_\tau)}}{e^{w(X_T-X_\tau)}}\right|\mathcal{F}_\tau\right]E_P\left[\left.e^{w(X_T-X_\tau)}\right|\mathcal{F}_\tau\right]=E_P\left[\left.e^{p(X_T-X_\tau)}\right|\mathcal{F}_\tau\right]$$
$$E_{P^{1-w}}\left[\left.\frac{e^{(1-p)(X_T-X_\tau)}}{e^{(1-w)(X_T-X_\tau)}}\right|\mathcal{F}_\tau\right]E_P\left[\left.e^{(1-w)(X_T-X_\tau)}\right|\mathcal{F}_\tau\right]=E_P\left[\left.e^{(1-p)(X_T-X_\tau)}\right|\mathcal{F}_\tau\right].$$

The self-duality property then follows by using the equality of the r.h.s. of the above equations and Lemma 3. ■

The following definition and proposition follow the unconditional versions stated in [15], see also [3].

Definition 6 *An adapted positive process S is **quasi self-dual** of order $\alpha\in\mathbb{R}$ if for any stopping time $\tau\le T$ and any non-negative Borel function f it holds that*

$$E_P\left[\left.f\left(\frac{S_T}{S_\tau}\right)\right|\mathcal{F}_\tau\right]=E_P\left[\left.\left(\frac{S_T}{S_\tau}\right)^\alpha f\left(\frac{S_\tau}{S_T}\right)\right|\mathcal{F}_\tau\right]. \tag{12}$$

In particular, for all $\tau\le T$

$$E_P\left[\left.\left(\frac{S_T}{S_\tau}\right)^\alpha\right|\mathcal{F}_\tau\right]=1.$$

Proposition 7 (Characterization of quasi self-duality) *S is quasi self-dual of order $\alpha\neq 0$ if and only if S^α is self-dual.*

Proof. This follows by considering for each f the functions g defined by $g(x)=f(x^\alpha)$, respectively h given by $h(x)=f\left(x^{1/\alpha}\right)$, $x>0$. ■

3 Quasi self-dual continuous martingales

The goal of this section is to clarify the structure of quasi self-dual processes in a continuous martingale setting which comprises some Brownian motion-driven stochastic volatility models in financial applications. Following [21] we assume throughout this section that every $(\mathcal{F}_t)$-martingale is continuous. We refer to [19] for all unexplained terminology.

For every continuous conditionally symmetric martingale Y, $Y_0 = 0$, such that its stochastic exponential $\mathcal{E}(Y)$ is a martingale, one can define the change of measure

$$\frac{dQ}{dP} = \mathcal{E}(Y)_T = \exp\left(Y_T - \frac{1}{2}[Y]_T\right).$$

In the sequel, we assume that Y is a continuous martingale with $Y_0 = 0$. Let $X = Y - \frac{1}{2}[Y]$ and observe that $[X] = [Y]$, hence $Y = X + \frac{1}{2}[X]$. We assume w.l.o.g. that $S_0 = 1$ and set $S = \exp(X) = \mathcal{E}(Y)$. By Proposition 5, the self-duality of a martingale S is equivalent to the conditional symmetry of X under the measure H. The next result is significantly more difficult to prove.

Theorem 8 (Tehranchi [21], Theorem 3.1.) *The continuous martingale S is self-dual if and only if S is of the form $S = \mathcal{E}(Y)$ for a conditionally symmetric continuous local martingale Y.*

One particular problem in this context is that stochastic exponentials can be strict local martingales in which case it would not be possible to use them as density processes for the measure transform leading to the dual market in financial interpretations. An example class of positive self-dual continuous martingales is provided by stochastic exponentials of conditionally symmetric BMO-martingales, see [13] for a detailed exposition of BMO-theory.

Proposition 9 *Let Y be a continuous conditionally symmetric martingale such that there exists a constant C with*

$$\sup_{0\leq\tau\leq T} E\left[|Y_T - Y_\tau| | \mathcal{F}_\tau\right] \leq C. \tag{13}$$

Then Y is a BMO-martingale and its stochastic Doléans-exponential $\mathcal{E}(\alpha Y)$ is a martingale for each $\alpha \in \mathbb{R}$. Moreover, the following two assertions are equivalent:

(i) *$S = \mathcal{E}(Y)$ is a positive self-dual martingale which satisfies for some $p > 1$ the reverse Hölder inequality $R_p(P)$, or, equivalently the Muckenhoupt inequality $A_q(Q)$ for $q = (p+1)/p$, both with the same constant.*

(ii) *Y is a conditionally symmetric BMO-martingale.*

Proof. Condition (13) implies that $Y \in BMO$. Consequently, by Theorem 2.3 in [13], $\mathcal{E}(\alpha Y)$ is a martingale (and not a strict local martingale).

The uniform boundedness for all stopping times of the l.h.s. of (5) for $f(x) = |x|^p$ corresponds to $R_p(P)$, and of the r.h.s. of (5) for $f(x) = |x|^q$ to $A_q(Q)$. The equivalence of (i) and (ii) then follows from Theorems 2.3, 2.4 and 3.4 in [13], together with Theorem 8. ■

The process $X = \log(S)$ is, in contrast to Y, typically not a martingale. As $X = Y - \frac{1}{2}[Y]$, the *minimal martingale measure* $\widehat{P}$ (see [20]) for X is well-defined if $\mathcal{E}\left(\frac{1}{2}Y\right)$ is a martingale, and has then the density

$$\frac{d\widehat{P}}{dP} = \exp(\frac{1}{2}Y_T - \frac{1}{8}[Y]_T) = \exp\left(\frac{1}{2}X_T + \frac{1}{8}[X]_T\right).$$

The *minimal entropy martingale measure* Q^E for X is a martingale measure which minimizes the relative entropy with respect to P over all martingale measures for X. It can be characterised as the martingale measure for X with finite relative entropy such that

$$\frac{dQ^E}{dP} = \exp\left(c + \int_0^T \eta_t \, dX_t\right),$$

where η is a predictable process with the property that $\int \eta \, dX$ is a Q-martingale for all martingale measures Q with finite relative entropy, see [11]. It follows from Corollary 5 that under mild conditions the measure H with density

$$\frac{dH}{dP} = \exp\left(c + \frac{1}{2}X_T\right)$$

is a martingale measure for X. The preceding discussion shows that typically, H is the minimal entropy martingale measure with $\eta = 1/2$. This is in general different from the minimal martingale measure, see [11], p. 1036. Moreover, it is remarkable that $Q^E = H$ has such a simple form, which has consequences for the structure of conditionally symmetric martingales. In fact, for all $t \in [0, T]$ the measure H^t with density

$$\frac{dH^t}{dP} = \exp\left(c_t + \frac{1}{2}X_t\right) \tag{14}$$

is a martingale measure for X on $[0, t]$ but the normalizing constant c_t depends of course on t.

Definition 10 *Let M be a continuous local martingale, and denote the right-continuous and complete filtration generated by M as $\mathbb{F}^M$. M is said to have the* ***PRP*** *(predictable representation property), if every $\mathbb{F}^M$-adapted local martingale N can be written as $N = N_0 + \int \vartheta \, dM$ for some predictable, M-integrable process ϑ.*

Proposition 11 *Let Y be a continuous P-martingale which is conditionally symmetric up to T and which has the PRP. Assume that $S = \mathcal{E}(Y)$ is a martingale, and that the minimal martingale measure $\widehat{P}$ exists for $X = Y - \frac{1}{2}[Y]$. Then Y is a Gaussian martingale.*

Proof. By [19], Exercise VIII 1.27., the fact that Y has the PRP under P implies that X has the PRP under $\widehat{P}$. Moreover, existence of $\widehat{P}$ implies existence of the probability measures H^t as defined in (14) because

$$E\left[\exp\left(\frac{1}{2}X_t\right)\right] \leq E\left[\exp\left(\frac{1}{2}X_t + \frac{1}{8}[X]_t\right)\right] = E\left[E\left[\left.\frac{d\widehat{P}}{dP}\right|\mathcal{F}_t\right]\right] = 1.$$

Since Y is conditionally symmetric up to T, it follows from Theorem 8 that S is self-dual, and hence, by Proposition 5, X is a conditionally symmetric martingale under each H^t. In particular, H^t is a martingale measure for X on $[0,t]$. The PRP implies by the second fundamental theorem of asset pricing, see Theorem 1.17 of [4], that $\widehat{P} = H^t$ on $\mathcal{F}_t$ which yields

$$\exp\left(\frac{1}{2}Y_t - \frac{1}{8}[Y]_t\right) = \exp\left(\frac{1}{2}X_t + \frac{1}{8}[X]_t\right) = \exp\left(c_t + \frac{1}{2}X_t\right),$$

for all $t \leq T$. It follows that $[Y] = [X]$ must be deterministic, and therefore Y is a Gaussian martingale. ■

The next result completely characterises quasi self-dual continuous semimartingales in terms of conditional symmetry.

Theorem 12 *A continuous positive semimartingale S is quasi self-dual of non-vanishing order $\alpha = 1 - 2\kappa$, if and only if S^α is a martingale and $S = e^{\kappa[M]}\mathcal{E}(M)$ for a continuous conditionally symmetric local martingale M. For $\alpha = 0$ we assume in addition that $S = \exp(M)$ for an integrable process M. In that case, S is quasi self-dual of order zero if and only if M is a continuous conditionally symmetric martingale.*

Proof. For $\alpha \neq 0$, S is quasi self-dual if and only if S^α is self-dual for some α, hence in particular a positive continuous martingale. We can then write by Theorem 8

$$S^\alpha = \mathcal{E}(\alpha M) = \exp\left(\alpha M - \frac{1}{2}\alpha^2[M]\right) = e^{\alpha\kappa[M]}\mathcal{E}(M)^\alpha$$

for some conditionally symmetric local martingale M. On the other hand, if $S = e^{\kappa[M]}\mathcal{E}(M)$, we have

$$S^{\alpha} = e^{\alpha\kappa[M]}\mathcal{E}(M)^{\alpha} = \exp\left(\alpha M + \left(\kappa - \frac{1}{2}\right)\alpha[M]\right).$$

Since S^{α} is a positive martingale it follows that $S^{\alpha} = \mathcal{E}(N)$ for a continuous local martingale N. The uniqueness of the canonical semimartingale decomposition implies $N = \alpha M$ which implies that

$$\left(\kappa - \frac{1}{2}\right)\alpha = -\frac{1}{2}\alpha^{2}.$$

In the case when $\alpha \neq 0$, dividing by α yields the result. If $\alpha = 0$, we start by assuming that S is quasi self-dual of order $\alpha = 0$. For an arbitrary non-negative Borel function f, define $g = f \circ \log$. By assumption we have for any stopping time $\tau \in [0, T]$

$$\begin{aligned} E\left[f\left(M_T - M_\tau\right)\middle|\,\mathcal{F}_\tau\right] &= E\left[g\left(\exp(M_T - M_\tau)\right)\middle|\,\mathcal{F}_\tau\right] \\ &= E\left[g\left(\exp(M_\tau - M_T)\right)\middle|\,\mathcal{F}_\tau\right] = E\left[f\left(M_\tau - M_T\right)\middle|\,\mathcal{F}_\tau\right], \end{aligned}$$

for an arbitrary non-negative Borel function f, i.e. M is conditionally symmetric combined with the integrability assumption, also a martingale, and it is clearly continuous. Furthermore, $S = \exp(M) = e^{\frac{1}{2}[M]}\mathcal{E}(M)$ holds.

Conversely, if $M = \log(S)$ is a continuous conditionally symmetric martingale, then for all non-negative Borel functions f define $g = f \circ \exp$ so that

$$\begin{aligned} E\left(f(\exp\left(M_T - M_\tau\right))\middle|\,\mathcal{F}_\tau\right) &= E\left(g\left(M_T - M_\tau\right)\middle|\,\mathcal{F}_\tau\right) \\ &= E\left(g\left(M_\tau - M_T\right)\middle|\,\mathcal{F}_\tau\right) = E\left(f\left(\exp(M_\tau - M_T)\right)\middle|\,\mathcal{F}_\tau\right), \end{aligned}$$

which implies the quasi self-duality of order zero. ■

Example 13 (Geometric Brownian motion) *The results presented lead to the following view of the symmetries of geometric Brownian motion. Let $Y = \sigma W$ for a standard Brownian motion W and $\sigma > 0$. Since Y is a continuous conditionally symmetric martingale, Theorem 8 yields that*

$$\mathcal{E}(Y) = \exp\left(Y - \frac{1}{2}[Y]\right) = \exp\left(\sigma W - \frac{1}{2}\sigma^{2}t\right)$$

is a self-dual process. Denoting by λ a shift parameter, we consider the process

$$S = \exp\left(\sigma W - \frac{1}{2}\sigma^2 t + \lambda t\right) = \exp\left(\kappa[Y]\right)\mathcal{E}\left(Y\right)\,, \quad \textit{where} \quad \kappa = \frac{\lambda}{\sigma^2}\,.$$

By Theorem 12 S is quasi self-dual of order zero if and only if $\lambda = \frac{1}{2}\sigma^2$ (since we have the ordinary exponent of a continuous conditionally symmetric martingale) and, in view of the fact that S^α, $\alpha = 1 - 2\kappa = 1 - \frac{2\lambda}{\sigma^2}$, is a martingale, it is quasi self-dual of order α, cf. e.g. [2, 3].

4 Ocone martingales and strong self-duality

In this section we discuss a connection between Ocone martingales and a strong version of self-duality of their associated stochastic exponentials, as motivated by the discussion in [21]. However, the previously introduced conditional notions of symmetry resp. self-duality are not quite fitting for such a discussion as Ocone martingales in particular enjoy a stronger notion of symmetry.

Definition 14 *Let M be a continuous P-martingale vanishing at zero and such that $[M]_\infty = \infty$, and consider its Dambis-Dubins-Schwartz (DDS) representation $M = B_{[M]}$. The process M is called an* ***Ocone martingale*** *if B and $[M]$ are independent.*

It has been proved in [22] that if a martingale is Ocone and has the PRP, then it is Gaussian. A more interesting example of an Ocone martingale is given by the solution of the stochastic differential equation

$$\begin{aligned} dM_t &= V_t\, dB_t, \\ dV_t &= -\mu V_t\, dt + \sqrt{V_t}\, dW_t, \end{aligned}$$

where $\mu > 0$ and B, W are two independent Brownian motions. This follows by [1], Ch. 2, Th. 2.6, since $[M] = \int V^2\, dt$ is independent of B. Moreover, Lévy's stochastic area process is also an Ocone martingale, see [22].

Definition 15 *An adapted process X is process symmetric if $X \sim -X$ (i.e. the finite dimensional distributions of X and $-X$ are the same). In particular, for semimartingales X with $X_0 = 0$ this is equivalent to*

$$E\left[\exp\left(i\int_0^T \theta_t\, dX_t\right)\right] = E\left[\exp\left(-i\int_0^T \theta_t\, dX_t\right)\right] \quad \forall \theta \in \mathcal{S},$$

where $\mathcal{S}$ denotes the space of deterministic and bounded Borel functions on $[0,T]$.

Remark 16 *Ocone martingales are always process symmetric, see Tehranchi [21].*

It is important to stress that different symmetry concepts are not equivalent. Since for example conditional symmetry implies the martingale property for integrable processes, we have that an integrable process symmetric X which is not a martingale cannot be conditionally symmetric. For example, if Z is a symmetric integrable random variable, then the process $(Zt)_{t\in[0,T]}$ is still process symmetric but not a martingale. Less obvious is that there are also process symmetric martingales which are not conditionally symmetric.

Example 17 (i) *The martingale $M = \int B^2\, dB$ is process symmetric since*

$$-\int B^2\, dB = \int (-B)^2\; d\,(-B)\,.$$

Since

$$B = \int \left(\frac{d[M]}{dt}\right)^{-\frac{1}{2}} dM\,,$$

we have that the Brownian filtration $\mathbb{F} = (\mathcal{F}_t)$ equals the filtration $\mathbb{F}^M$ generated by M. Moreover, M has the PRP, but is non-Gaussian, and hence not Ocone.

(ii) *It is worth noting, in light of Theorem 22, that the stochastic exponential $\mathcal{E}\,(M)$ is a strict local martingale. This follows e.g. by Corollary 2.2 of [14] since the with M associated auxiliary diffusion*

$$d\widetilde{Y}_t = \widetilde{Y}_t^2\, dt + dB_t$$

does explode.

(iii) *However, M is not conditionally symmetric. Choose $0 < t < T$ and assume by means of contradiction that M is conditionally symmetric. In particular,*

$$E[(M_T - M_t)^3|\mathcal{F}_t] = E[(M_t - M_T)^3|\mathcal{F}_t]\,,$$

since the symmetry is satisfied by the positive and the negative part of the third conditional moment, so that

$$E[(M_T - M_t)^3|\mathcal{F}_t] = 0$$

holds a.s., where we have used that $E\left[|M_T - M_t|^3\right] < \infty$. However, noting that $W_s = B_{t+s} - B_t$ defines a Brownian motion independent of $\mathcal{F}_t$ and that B_t is $\mathcal{F}_t$-measurable, we can write

$$M_T - M_t = \int_0^{T-t} W_s dW_s + B_t^2 W_{T-t} + B_t(W_{T-t}^2 - (T-t))\,.$$

By a straightforward but lengthy calculation, the third conditional moment of this martingale increment can, for $t < T$, be written as a nontrivial polynomial in B_t, which, for $t > 0$, will not take a.s. only values in the roots of the polynomial, i.e. we end up with a contradiction.

It is observed by Tehranchi [21] that continuous Ocone martingales are conditionally symmetric with respect to deterministic times. The next result shows that this is still true for bounded stopping times.

Lemma 18 *A continuous Ocone martingale M with natural filtration $\mathbb{F} = (\mathcal{F}_t)$ is conditionally symmetric.*

Proof. Since M is Ocone we have $M = \beta_{[M]}$ for a Brownian motion β (with natural filtration $\mathbb{B} = (\mathcal{B}_t)$) being independent of $[M]$. Denote by $\mathbb{G} = (\mathcal{G}_t)$ the right-continuous enlargement of the filtration $\mathcal{B}_t \vee \sigma([M])$. Following [5, p. 129] we have that for τ being an $\mathbb{F}$-stopping time it follows that $[M]_\tau$ is a $\mathbb{G}$-stopping time. Indeed, by denoting $A_t = \inf\{s : [M]_s > t\}$ we have

$$\{[M]_\tau \le t\} = \{\tau \le A_t\} \in \mathcal{F}_{A_t} \subset \bigcap_{\varepsilon>0} \sigma(M^{A_{t+\varepsilon}}) \subset \bigcap_{\varepsilon>0} [\mathcal{B}_{t+\varepsilon} \vee \sigma([M])] = \mathcal{G}_t\,,$$

where in our case $\tau \le T < \infty$ and $[M]$ is continuous, so that we end up with a finite stopping time. As in the proof of Lemma 2 on p. 129 in [5] the Ocone property and Lemma 1 on p. 129 in [5] imply that β is a $\mathbb{G}$-Brownian motion. By [12, Th. 13.11] $\beta'_u = \beta_{u+[M]_\tau} - \beta_{[M]_\tau}$ is a Brownian motion independent of $\mathcal{G}_{[M]_\tau}$. Hence, for $\tau \le T < \infty$ we have for any non-negative Borel function f

$$\begin{aligned} E[f(M_T - M_\tau)|\mathcal{F}_\tau] &= E[E[f(\beta_{[M]_T} - \beta_{[M]_\tau})|\mathcal{G}_{[M]_\tau}]|\mathcal{F}_\tau] \\ &= E[E[f(\beta_{[M]_\tau} - \beta_{[M]_T})|\mathcal{G}_{[M]_\tau}]|\mathcal{F}_\tau] = E[f(M_\tau - M_T)|\mathcal{F}_\tau]\,, \end{aligned}$$

so that M is conditionally symmetric. ■

We assume w.l.o.g. that $S_0 = 1$ and set $S = \exp(X) = \mathcal{E}(Y)$ where X (and then Y) is a continuous semimartingale.

Recall that $\mathcal{S}$ is the space of deterministic and bounded Borel functions on $[0, T]$. For $\phi \in \mathcal{S}$, we set $X^\phi = \int \phi\, dY - \frac{1}{2}\int \phi^2\, d[Y]$, and $S^\phi := \exp\left(X^\phi\right) = \mathcal{E}\left(\int \phi\, dY\right)$. In the case that all S^ϕ are martingales, we define dual probability measures Q^ϕ via

$$\frac{dQ^\phi}{dP} = S^\phi_T.$$

Definition 19 *We say that $S = \mathcal{E}(Y)$, for a continuous martingale Y, is strongly self-dual if for all $\phi \in \mathcal{S}$, the S^{ϕ} are martingales and that with equality in distribution as a process living on $[0,T]$,*

$$\left\{S^{\phi}, P\right\} \sim \left\{\frac{1}{S^{\phi}}, Q^{\phi}\right\} .$$

In the case when S is strongly self-dual then, by choosing $\phi = 1$, it also implies an unconditional form of the self-duality property known as put-call symmetry, which is the most frequently used definition in the previous literature.

Lemma 20 *S is strongly self-dual if and only if for all $\phi \in \mathcal{S}$, the S^{ϕ} are martingales and*

$$\left\{X^{\phi}, P\right\} \sim \left\{-X^{\phi}, Q^{\phi}\right\} . \tag{15}$$

Proof. Note that the martingale assumptions are the same. Assume that S is strongly self-dual. For an arbitrary non-negative functional F define G via

$$G := F \circ \log .$$

Then

$$\begin{aligned} E_P\left[F(X_t^{\phi}, 0 \le t \le T)\right] &= E_P\left[G(S_t^{\phi}, 0 \le t \le T)\right] = E_{Q^{\phi}}\left[G\left(\frac{1}{S_t^{\phi}}, 0 \le t \le T\right)\right] \\ &= E_{Q^{\phi}}\left[F(-X_t^{\phi}, 0 \le t \le T)\right] , \end{aligned}$$

so that (15) follows. The converse direction follows similarly. ■

For the record, we state the next proposition which is analogous to Corollary 5. If for $\phi \in \mathcal{S}$, $\exp\left(X^{\phi}\right)$ is a P-martingale, then

$$c_{\phi} = E\left[\exp\left(\frac{1}{2}X_T^{\phi}\right)\right] < \infty,$$

so we can define probability measures H^{ϕ} (analogous to the half measure H) via

$$\frac{dH^{\phi}}{dP} = c_{\phi}^{-1} \exp\left(\frac{1}{2}X_T^{\phi}\right).$$

Proposition 21 *S is strongly self-dual if and only if for all $\phi \in \mathcal{S}$ we have that the S^{ϕ} are martingales and the X^{ϕ} are process symmetric under the measures H^{ϕ}.*

Proof. The strong self-duality implies the martingale assumptions and with Lemma 20 it implies for arbitrary λ, $\phi \in \mathcal{S}$ that

$$c_\phi^{-1} E_P \left[\exp \left(\frac{1}{2} X_T + i \int_0^T \lambda_t \, dX_t^\phi \right) \right] = c_\phi^{-1} E_{Q^\phi} \left[\exp \left(\frac{1}{2}(-X_T) + i \int_0^T \lambda_t \, d(-X_t^\phi) \right) \right] ,$$

while from the definitions of H^ϕ and Q^ϕ

$$E_{H^\phi} \left[\exp \left(i \int_0^T \lambda_t \, dX_t^\phi \right) \right] = c_\phi^{-1} E_P \left[\exp \left(\frac{1}{2} X_T + i \int_0^T \lambda_t \, dX_t^\phi \right) \right]$$

and

$$\begin{aligned} E_{H^\phi} \left[\exp \left(-i \int_0^T \lambda_t \, dX_t^\phi \right) \right] &= c_\phi^{-1} E_P \left[\exp \left(\frac{1}{2} X_T + i \int_0^T \lambda_t \, d(-X_t^\phi) \right) \right] \\ &= c_\phi^{-1} E_{Q^\phi} \left[\exp \left(\frac{1}{2}(-X_T) + i \int_0^T \lambda_t \, d(-X_t^\phi) \right) \right] . \end{aligned}$$

Since λ and ϕ were arbitrarily chosen, we end up with the process symmetries of the processes X^ϕ under H^ϕ.

Conversely, the process symmetries imply for arbitrary λ (and ϕ)$\in \mathcal{S}$ that

$$c_\phi E_{H^\phi} \left[\exp \left(-\frac{1}{2} X_T + i \int_0^T \lambda_t \, dX_t^\phi \right) \right] = c_\phi E_{H^\phi} \left[\exp \left(-\frac{1}{2}(-X_T) + i \int_0^T \lambda_t \, d(-X_t^\phi) \right) \right] ,$$

while again by the definitions of H^ϕ and Q^ϕ

$$E_P \left[\exp \left(i \int_0^T \lambda_t \, dX_t^\phi \right) \right] = c_\phi E_{H^\phi} \left[\exp \left(-\frac{1}{2} X_T + i \int_0^T \lambda_t \, dX_t^\phi \right) \right]$$

and

$$\begin{aligned} E_{Q^\phi} \left[\exp \left(-i \int_0^T \lambda_t \, dX_t^\phi \right) \right] &= E_P \left[\exp \left(X_T + i \int_0^T \lambda_t \, d(-X_t^\phi) \right) \right] \\ &= c_\phi E_{H^\phi} \left[\exp \left(-\frac{1}{2}(-X_T) + i \int_0^T \lambda_t \, d(-X_t^\phi) \right) \right] . \end{aligned}$$

Hence, in view of the imposed martingale assumptions, the strong self-duality now follows by Lemma 20. ∎

In the following result we show that the Ocone property translates one-to-one via lifting by stochastic exponentiation into the strong self-duality property.

Theorem 22 *A continuous martingale Y is an Ocone martingale if and only if $\mathcal{E}(Y)$ is strongly self-dual.*

Proof. Let first Y be a continuous Ocone martingale, and set for $\phi \in \mathcal{S}$

$$\widetilde{Y} = Y - \int \phi \, d[Y] .$$

Here we suppress the dependency of $\widetilde{Y}$ on ϕ for ease of notation.

By Theorem 1 and Comment 2 of Vostrikova & Yor [22], it holds that Y is a continuous Ocone martingale if and only if

(i) For all $\phi \in \mathcal{S}$,

$$\{Y , P\} \sim \left\{\widetilde{Y} , Q^{\phi}\right\} , \tag{16}$$

or equivalently,

$$\{[Y] , P\} \sim \left\{[Y] , Q^{\phi}\right\} ; \tag{17}$$

note that $[Y] = \left[\widetilde{Y}\right]$.

(ii) $\mathcal{E}\left(\int \phi \, dY\right)$ is a martingale for all $\phi \in \mathcal{S}$.

We now show that $\left\{X^{\phi} , P\right\} \sim \left\{-X^{\phi} , Q^{\phi}\right\}$ for all $\phi \in \mathcal{S}$, which implies (in view of (ii)) by Lemma 20 strong self-duality of $S = \mathcal{E}(Y)$. First note that $\tilde{Y}$ under Q^{ϕ} is also Ocone and thus, in particular process symmetric. Furthermore, we have for all $\lambda , \psi \in \mathcal{S}$ by the aforementioned properties of Ocone martingales, cf. also Lemma 2.5 of [17], that

$$\begin{aligned}
&E\left[\exp\left(i \int_0^T \lambda_t \, dX_t^{\phi}\right)\right] = E\left[\exp\left(i \int_0^T \lambda_t \, d\left(\int_0^t \phi_s \, dY_s - \frac{1}{2}\int_0^t \phi_s^2 \, d[Y]_s\right)\right)\right] \\
&= E_{Q^{\phi}}\left[\exp\left(i \int_0^T \lambda_t \, d\left(\int_0^t \phi_s \, d\widetilde{Y}_s - \frac{1}{2}\int_0^t \phi_s^2 \, d\left[\widetilde{Y}\right]_s\right)\right)\right] \\
&= E_{Q^{\phi}}\left[\exp\left(i \int_0^T \lambda_t \, d\left(\int_0^t \phi_s \, d\left(-\widetilde{Y}_s\right) - \frac{1}{2}\int_0^t \phi_s^2 \, d\left[\widetilde{Y}\right]_s\right)\right)\right] \\
&= E_{Q^{\phi}}\left[\exp\left(i \int_0^T \lambda_t \, d\left(-\left(\int_0^t \phi_s \, dY_s - \int_0^t \phi_s^2 \, d[Y]_s + \frac{1}{2}\int_0^t \phi_s^2 \, d[Y]_s\right)\right)\right)\right] \\
&= E_{Q^{\phi}}\left[\exp\left(i \int_0^T \lambda_t \, d\left(-\left(\int_0^t \phi_s \, dY_s - \frac{1}{2}\int_0^t \phi_s^2 \, d[Y]_s\right)\right)\right)\right] \\
&= E_{Q^{\phi}}\left[\exp\left(-i \int_0^T \lambda_t \, dX_t^{\phi}\right)\right]
\end{aligned}$$

which proves the claim and hence the first implication.

As for the other direction, we will show property (17) for all $\phi \in \mathcal{S}$. Let $\mathcal{E}\left(\int \phi\, dY\right) = \exp(X^{\phi})$ where $X^{\phi} = \int \phi\, dY - \frac{1}{2}\int \phi^2\, d[Y]$, and define probability measures Q^{ϕ} via

$$dQ^{\phi}/dP = \mathcal{E}\left(\int \phi\, dY\right)_T = \exp\left(X_T^{\phi}\right).$$

We assume first that ϕ is bounded away from zero, and that Y is a square-integrable martingale, i.e. $E\left[Y_t^2\right] < \infty$ for all $t \geq 0$. Note that $\left[X^{\phi}\right] = \int \phi^2\, d[Y]$ and therefore $[Y] = \int \phi^{-2}\, d\left[X^{\phi}\right]$. We have for every non-negative functional F, with $F(U)$ denoting $F(U_t; 0 \leq t \leq T)$ for any stochastic process U,

$$E_{Q^{\phi}}\left[F([Y])\right] = E_P\left[\exp\left(X_T^{\phi}\right) F\left(\int \phi^{-2}\, d\left[X^{\phi}\right]\right)\right] = E_{Q^{\phi}}\left[F\left(\int \phi^{-2}\, d\left[X^{\phi}\right]\right)\right].$$

On the other hand,

$$E_P\left[F([Y])\right] = E_P\left[F\left(\int \phi^{-2}\, d\left[X^{\phi}\right]\right)\right].$$

Hence the strong self-duality of S implies by Lemma 20 that $[Y]$ under Q^{ϕ} has the same law as $[Y]$ under P, for all $\phi \in \mathcal{S}$ which are bounded away from zero. Equivalently, for all such ϕ; $N \in \mathbb{N}$ arbitrary; $0 \leq t_1 \leq ... \leq t_N \leq T$; and arbitrary $u = (u_1, ..., u_N) \in \mathbb{R}^N$ we set

$$\Psi(u) = \exp\left(i\left(u_1 [Y]_{t_1} + ... + u_N [Y]_{t_N}\right)\right),$$

and have that

$$\begin{aligned} E\left[\Psi(u) \exp\left(\int_0^T \phi_t\, dY_t - \frac{1}{2}\int_0^T \phi_t^2\, d[Y]_t\right)\right] &= E_{Q^{\phi}}\left[\Psi(u)\right] \\ &= E\left[\Psi(u)\right]. \end{aligned} \tag{18}$$

Let now $\phi \in \mathcal{S}$ be arbitrary, i.e., in particular ϕ may vanish on a set $\Gamma \subset [0, T]$. We denote by $\phi^{(n)} \in \mathcal{S}$ functions which coincide with ϕ as long as $|\phi_t| \geq 1/n$, and which equal $1/n$ if $|\phi_t| \leq 1/n$, so that $\phi^{(n)} \to \phi$ pointwise. By dominated convergence for stochastic integrals (see [19], Theorem IV.2.12), it follows that in probability,

$$U_n := \mathcal{E}\left(\int \phi^{(n)}\, dY\right)_T \to \mathcal{E}\left(\int \phi\, dY\right)_T.$$

We will now show that the family $\{U_n\}_{n\in\mathbb{N}}$ is uniformly integrable. For this, it suffices to show that

$$\sup_n E\left[U_n \log\left(U_n\right)\right] < \infty.$$

Indeed, using that so far Y was assumed to be a square-integrable martingale, and therefore $E\left[[Y]_t\right] < \infty$ for all $t \geq 0$ (see [18], Corollary II.6.3),

$$\begin{aligned}
E\left[U_n \log\left(U_n\right)\right] &= E_{Q^{\phi^{(n)}}}\left[\int_0^T \phi_t^{(n)}\, dY_t - \frac{1}{2}\int_0^T \left(\phi_t^{(n)}\right)^2 d\left[Y\right]_t\right] \\
&= E_{Q^{\phi^{(n)}}}\left[\int_0^T \phi_t^{(n)}\, dY_t - \int_0^T \left(\phi_t^{(n)}\right)^2 d\left[Y\right]_t + \frac{1}{2}\int_0^T \left(\phi_t^{(n)}\right)^2 d\left[Y\right]_t\right] \\
&= E_{Q^{\phi^{(n)}}}\left[\frac{1}{2}\int_0^T \left(\phi_t^{(n)}\right)^2 d\left[Y\right]_t\right] \\
&= E\left[\frac{1}{2}\int_0^T \left(\phi_t^{(n)}\right)^2 d\left[Y\right]_t\right],
\end{aligned}$$

since by what has already been proved, $[Y]$ has the same distribution under both $Q^{\phi^{(n)}}$ and P since the $\phi^{(n)}$ are bounded away from zero. We have, since Y was assumed to be square-integrable,

$$E\left[\frac{1}{2}\int_0^T \left(\phi_t^{(n)}\right)^2 d\left[Y\right]_t\right] \leq \text{const.} E\left[[Y]_T\right] < \infty.$$

Hence the U_n are uniformly integrable, and we conclude by (18) that

$$\begin{aligned}
&E\left[\Psi(u)\exp\left(\int_0^T \phi_t\, dY_t - \frac{1}{2}\int_0^T \phi_t^2\, d\left[Y\right]_t\right)\right] \\
&= \lim_n E\left[\Psi(u)\exp\left(\int_0^T \phi_t^{(n)}\, dY_t - \frac{1}{2}\int_0^T \left(\phi_t^{(n)}\right)^2 d\left[Y\right]_t\right)\right] \\
&= E\left[\Psi(u)\right].
\end{aligned}$$

Therefore $[Y]$ has the same distribution under both Q^{ϕ} and P for all $\phi \in \mathcal{S}$. It follows by the aforementioned result of [22] that Y is an Ocone martingale.

In the case that Y is a continuous martingale, not necessarily square-integrable, the result follows by localization: there is an increasing sequence of stopping times (T_n) such that Y^{T_n} is bounded, e.g. $T_n = \inf\{t : |Y_t| = n\}$. Therefore the previous result applies on $[0, T_n]$, each n. Letting n tend to infinity yields then the general result. ■

This result should be seen in the context of Tehranchi's [21] Theorem as given in Theorem 8. In both results, a certain symmetry property (conditional respectively Ocone symmetry) translates into a self-duality property (conditional respectively strong) of the associated stochastic exponential. The structure of the proofs, however, is completely different. While Tehranchi's proof rests on his characterization that for a continuous local martingale Y conditional symmetry is equivalent to the property that Y_T given $\mathcal{F}_t \vee \sigma\left([Y]_T\right)$ is normally distributed with expectation Y_t and variance $[Y]_T - [Y]_t$ for all $0 \leq t \leq T$, we work with properties of Ocone martingales as developed in [17] and [22]. Again, it is still open whether there exists *any* non-Ocone conditionally symmetric martingale.

The notion of strong self-duality is justified by an economic interpretation, namely that the distribution of the price process S^{ϕ} for arbitrary parameter function $\phi \in \mathcal{S}$ remains invariant under the dual market transformation. Furthermore, it is shown in Theorem 22 that it is equivalent to the Ocone property of its stochastic logarithm. The relationsship between conditional and strong self-duality is beyond the scope of this paper.

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