Static replication of barrier options: some general results

Leif B. G. Andersen
Managing Director, Banc of America Securities, 9 West 57th Street, New York, NY 10019, USA

Jesper Andreasen
Director, Nordea Markets, Strandgade 3, Copenhagen, 0900 C, Denmark

David Eliezer
c/o L.B.G. Andersen, Managing Director, Banc of America Securities, 9 West 57th Street, New York, NY 10019, USA

This paper presents a number of new theoretical results for the replication of barrier options through a static portfolio of European put and call options. Our results are valid for options with completely general knock-out/knock-in sets, and allow for time- and state-dependent volatility as well as discontinuous asset dynamics. We illustrate the theory with numerical examples and discuss practical implementation.

1 Introduction

The classical approach to the hedging of derivatives involves maintaining an ever-changing position in the underlying assets. The construction of such *dynamic hedges* is a key argument in the seminal paper by Black and Scholes (1973), and is a standard technique for practical hedging of derivative products. A literal interpretation of dynamic hedging strategies, however, requires continuous trading, which would generate enormous transaction costs if implemented in practice. Instead, most real-life trading strategies involve time-discrete rebalancing, exposing the hedger to some risk, particularly if the gamma of the option hedged is high.

For some derivatives, it turns out that it is possible to construct a hedge that does not involve continuous rebalancing. Such *static hedges* normally involve setting up a portfolio of simple, European options (typically puts and calls) that is guaranteed to match the payoff of the instrument to be hedged. It is fair to say that less is known about static hedging than dynamic hedging, although recent papers have made some progress. Derman, Ergener and Kani (1995) describe a numerical algorithm for single barrier options in the context of a binomial tree representing the evolution of a stock with time- and level-dependent volatility. Bowie and Carr (1994), Carr and Chou (1997) and Carr, Ellis and Gupta (1998) examine in detail the static replication of barrier options in the Black–Scholes

This paper was completed while all three authors were employed by General Re Financial Products.

(1973) model. For martingale stock processes, Brown, Hobson and Rogers (1998) demonstrate how to set up model-free over- and underhedges for certain simple classes of single-barrier options.

The approach in this paper extends the results in previous literature in a number of ways. First and foremost, we derive exact, explicit expressions for the composition of statically replicating portfolios in asset models that allow for both jumps and time- and state-dependent diffusion volatility. Second, we are able to derive static hedging portfolios not only for simple, continuously monitored barrier options but we also allow for almost arbitrarily complicated knock-out regions and terminal payoffs (and can easily handle curved, discrete, partial, and double-barrier options).

All of our theoretical results are derived under the assumption (or approximation) that European options are traded in inelastic supply for all maturities and strikes. This is not true in practice and we therefore devote a section of the paper to treating some issues that arise in the practical implementation of the static hedging strategies suggested in the first part of the paper.

The rest of the paper is organized as follows: Section 2 derives static hedges for general barrier options written on an asset with a volatility that depends deterministically on time and the asset itself. In Section 3 we extend our results to the case of discontinuous asset dynamics. Section 4 investigates some issues relating to the practical implementation of static hedging strategies and presents numerical results. Finally, Section 5 contains the conclusions of the paper. The first of two appendices demonstrates how the results in the paper – which are derived using probabilistic techniques – can alternatively be proven by the more traditional tools of differential forms and circulation theorems. The second appendix briefly considers the case of stochastic volatility and demonstrates that our technique does not lead to a static hedge for this case.

Finally, let us point out that the study at hand is largely applied in nature. As our main focus is new formulas and the ideas behind them, we have de-emphasized technicalities and set the paper in a relatively loose mathematical frame. In particular, we have put little emphasis in the specification of technical regularity conditions, which we trust that readers can supply themselves.

2 Deterministic volatility

In this section we derive static hedging portfolios for barrier options written on an underlying stock (or foreign exchange rate) characterized by a local volatility that is only a function of time and stock price level. Such asset price dynamics are discussed in detail in Dupire (1994). For ease of notation we make the simplifying assumption that all interest rates and dividend yields are zero.[1] Next we assume that the underlying stock (or foreign exchange rate) evolves according to

$$\frac{\mathrm{d}S(t)}{S(t)} = \sigma\,(t, S(t))\,\mathrm{d}W(t) \qquad (1)$$

where σ is a continuous, deterministic function, and W is a Brownian motion under the risk-neutral measure. We assume that σ is positive and sufficiently regular for (1) to have a unique, non-explosive, positive solution.

We further assume that we can trade European options on the stock with all maturities and strikes. We will let $C(T,K)$ and $P(T,K)$ denote the time-0 prices of European call and put options, respectively, with maturity T and strike K. We let $C(t;T,K)$ and $P(t;T,K)$ denote the same options' prices at time t. We note that European option prices are linked to the risk-neutral marginal density of the stock price. Specifically, if we let $f(T,S)$ denote the time-0 marginal density of $S(T)$ taken in S, we have that

$$f(T,S) \equiv E\left[\delta(S(T)-S)\right] = C_{KK}(T,S) = P_{KK}(T,S)$$

$$\int_B^\infty f(T,S)\,\mathrm{d}S = E\left[1_{S(T)\geq B}\right] = -C_K(T,B) = 1-P_K(T,B)$$

$$\int_0^B f(T,S)\,\mathrm{d}S = E\left[1_{S(T)\leq B}\right] = 1+C_K(T,B) = P_K(T,B) \tag{2}$$

where subscripts denote partial derivatives, $\delta(\cdot)$ is Dirac's delta function, $\mathrm{E}(\cdot)$ is the time-0 risk-neutral expectations operator, and 1_A denotes the indicator function on the set A.

2.1 Continuous barriers

Consider the function $F = F(t,S)$, defined as the solution to

$$\begin{aligned} F_t(t,S) + \frac{1}{2}\sigma(t,S)^2 S^2 F_{SS}(t,S) &= 0, \qquad t<T,\ S>B(t) \\ F(t,S) &= R(t), \qquad t<T,\ S\leq B(t) \\ F(T,S) &= g(S), \qquad \forall S \end{aligned} \tag{3}$$

where g is a function of the stock price only, and B is a continuous function of time on $[0,T]$. We recognize (3) as the PDE formulation of the problem of pricing a down-and-out barrier option with time-dependent rebate $R(t)$ and time-dependent continuously observed barrier level, $B(t)$. Here, and throughout the paper, we assume that R is a differentiable function.[2] Note that we let g define the terminal value of $F(T,S)$ for all values of S, including the knock-out region $S \leq B(T)$. So, if for example we consider a down-and-out call option, then

$$g(S) = (S-K)^+ 1_{S>B(T)} + R(T)\,1_{S\leq B(T)}$$

It should be stressed that $F(t)$ is the value of a barrier option *initiated* at time t, ie, if $G(t;s)$ is the time-t value of a barrier option originally issued at time $s \leq t$, then $F(t) = G(t;t)$. This means that F, unlike G, is not a martingale under the risk-neutral measure (as will be evident shortly). If we know that S did not breach the barrier in $[0, t]$, then $F(t) = G(t; 0)$; this relation obviously only holds up to the first time S hits the barrier.

Using (3) and the fact that F is continuous, but not generally continuous-differentiable, at $S = B$ we get from Tanaka's formula (Karatzas and Shreve, 1991) and (3) that

$$\mathrm{d}F(t, S(t)) = 1_{S>B(t)} F_S(t, S(t))\, S(t)\, \sigma(t, S(t))\, \mathrm{d}W(t) + 1_{S<B(t)} R'(t)\, \mathrm{d}t$$
$$+ \frac{1}{2} \sigma(t, S(t))^2 B(t)^2 F_S(t, B(t)+)\, \delta(S(t) - B(t))\, \mathrm{d}t \tag{4}$$

where $R' = \mathrm{d}R/\mathrm{d}t$ is assumed to exist, and $F_S(t, B(t)+)$ is the limit of $F_S(t, B(t) + \varepsilon)$ for $\varepsilon \downarrow 0$.

Integrating (4) in the time-dimension yields

$$g(S(T)) - F(0, S(0)) = \int_0^T 1_{S(t)>B(t)} F_S(t, S(t))\, \sigma(t, S(t))\, S(t)\, \mathrm{d}W(t)$$
$$+ \int_0^T 1_{S(t)<B(t)} R'(t)\, \mathrm{d}t$$
$$+ \frac{1}{2} \int_0^T F_S(t, B(t)+)\, \sigma(t, B(t))^2 B(t)^2\, \delta(S(t) - B(t))\, \mathrm{d}t$$

Taking expectations and rearranging yields the relation

$$F(0, S(0)) = \mathrm{E}\left[g(S(T))\right] - \int_0^T R'(t)\, \mathrm{E}\left[1_{S(t)<B(t)}\right] \mathrm{d}t$$
$$- \frac{1}{2} \int_0^T \sigma(t, B(t))^2 B(t)^2 f(t, B(t))\, F_S(t, B(t)+)\, \mathrm{d}t$$

Notice that we have here used the fact that F_S is a deterministic function around the barrier; were the stock volatility stochastic, this would *not* hold.[3] The formula above relates the barrier option price to the volatility and the (risk-neutral) marginal density. Interestingly, the first passage-time densities and conditional probabilities are not directly involved here.[4] The marginal density can be synthesized using option positions by use of (2). We get

$$F(0, S(0)) = \int_0^\infty g(S) P_{KK}(T, S)\, \mathrm{d}S - \int_0^T \int_0^{B(t)} R'(t) P_{KK}(t, S)\, \mathrm{d}S\, \mathrm{d}t$$
$$- \frac{1}{2} \int_0^T \sigma(t, B(t))^2 B(t)^2 F_S(t, B(t)+) P_{KK}(t, B(t))\, \mathrm{d}t \tag{5}$$

where we have arbitrarily chosen to synthesize the density from put options. We note that (5) expresses the value of a barrier option as a linear combination of puts, specifically:

- ❏ long a continuum $\{g(S)\}_{0<S<\infty}$ of T-maturity butterfly put spreads $P_{KK}(T,S)$;
- ❏ short a double continuum $\{R'(t)\}_{0<t<T}$ of butterfly put spreads $P_{KK}(t,S)$ with strikes in $[0,B(t)]$; and
- ❏ short a continuum $\{\sigma(t,B(t))^2 B(t)^2 F_S(t,B(t)+)\}_{0<t<T}$ of butterfly spreads with strikes along the barrier.

Consider now using the put portfolio suggested by (5) as a hedge for a barrier option $G(t;0)$ initiated at time 0. Specifically, if $\tau = \inf\{t\colon S(t) = B(t)\}$ is the first time the stock touches the barrier, we hold the put portfolio up to $\tau \vee T$ and, if $\tau < T$, sell off the outstanding portfolio at the time the barrier is breached. As mentioned earlier, $F(t) = G(t;0)$ up to (and including) the minimum of τ and T, whereby such a strategy would clearly generate the correct cashflow at $\tau \vee T$. For the put portfolio to qualify as a static hedge, we need to verify that the portfolio does not generate any other cashflows at times $t < \tau \vee T$. But as all the put positions with maturities less than T only involve strikes at or below the barrier, clearly no such cashflows are generated; whence, the put portfolio in (5) *qualifies as a static hedge*.

Although (5) is a static hedge, it is not necessarily the most convenient. In particular, we notice that the second term in (5) can be simplified to

$$\int_0^T \int_0^{B(t)} R'(t)\, P_{KK}(t,S)\, \mathrm{d}S\, \mathrm{d}t = \int_0^T R'(t)\, P_K(t, B(t))\, \mathrm{d}t$$

which represents a position of put spreads along the barrier. This position does not generate cashflows before the option matures or knocks out,[5] and the hedge remains static. We can simplify the hedge even further by relating the butterfly spreads to calendar spreads through the forward equations of Dupire (1994):

$$0 = -C_T + \frac{1}{2}\sigma(T,K)^2 K^2 C_{KK}; \qquad 0 = -P_T + \frac{1}{2}\sigma(T,K)^2 K^2 P_{KK}$$

We can now rewrite (5) as simply

$$F(0, S(0)) = \int_0^\infty g(S)\, P_{KK}(T,S)\, \mathrm{d}S - \int_0^T F_S\big(t, B(t)+\big)\, P_T\big(t, B(t)\big)\, \mathrm{d}t - \int_0^T R'(t)\, P_K\big(t, B(t)\big)\, \mathrm{d}t \tag{6}$$

As calendar put spreads on the barrier do not produce cashflows as long as the barrier option is "alive", (6) represents a static hedge, where the barrier option is now replicated by a European option paying g at maturity, minus the (deterministic) continuum $\{F_S(t,B(t)+)\}_{0<t<T}$ of calendar spreads along the barrier, and minus a continuum $\{R'(t)\}_{0<t<T}$ of put spreads along the barrier. As mentioned earlier, the options positions must be unwound when the barrier is hit. If the model is correct – ie, if the delta (F_S) along the barrier of alive options is computed correctly – then the unwind gain equals the rebate.

As written in (6), hedging the European payoff paying g is accomplished through butterfly spreads. Alternatively, we assemble the European payoff directly from the "hockey-stick" building blocks of puts and calls. Following Carr and Chou (1997), this can be accomplished by writing

$$g(S) = g(\kappa) + g'(\kappa)(S-\kappa) + \int_0^{\kappa} g''(K)\,[K-S]^+\,\mathrm{d}K + \int_{\kappa}^{\infty} g''(K)\,[S-K]^+\,\mathrm{d}K$$

for some arbitrary positive constant κ. Setting $\kappa = B(T)-$ and integrating over the density of S yields

$$\begin{aligned} E[g(S(T))] = {} & R(T)\,P_K(T, B(T)) \\ & + \int_{B(T)+}^{\infty} g''(K)\,C(T,K)\,\mathrm{d}K \\ & + g'(B(T)+)\,C(T, B(T)) - C_K(T, B(T))\,g(B(T)+) \end{aligned} \tag{7}$$

(7) represents a static hedge consisting of a continuum of calls with strikes above the barrier, plus a finite number of calls and put–call spreads with strikes at the barrier. Notice that if g has kinks or discontinuities, the derivatives of g in (7) must, of course, be interpreted in the sense of distributions.

It is worth noting that (6)–(7) only require model-based computation of the delta along the barrier – for instance by a finite-difference scheme (see, eg, Andersen and Brotherton-Ratcliffe (1998) for a discussion of the implementation of the dynamics (1) in a finite-difference scheme); all other terms in the hedging portfolio can be deduced from the market prices of standard European options.

The technique outlined above is easy to apply to many types of barrier options, including "in"-style barrier options. Sometimes we can also rely on parity results; for instance, a down-and-in option can be written as a European option minus a down-and-out option (with no rebate), whereby the results derived above can be used directly to statically hedge a down-and-in option. Applications to double-barrier options are simple as well, and would merely involve including in (6) an extra integral of call maturity-spreads and an extra integral of call spreads along the second barrier.[6] We will return to more general barrier shapes in a later section.

2.2 Discretely monitored barriers

Consider now the case when the down-and-out barrier of the previous section is only monitored on a discrete set of dates:

$$0 \le t_0 < \ldots < t_n < T$$

The PDE formulation of the pricing problem is

$$F_t(t,S)+\frac{1}{2}\sigma(t,S)^2S^2F_{SS}(t,S)=0, \qquad (t,S)\notin\{(t,S)\,|\,t\in\{t_i\},\ S\le B(t_i)\}$$

$$F(t_i,S)=R(t_i), \quad S\le B(t_i)$$

$$F(T,S)=g(S), \quad \forall S \tag{8}$$

Since the option price is discontinuous in the time dimension across every barrier time t_i for all $S \le B(t_i)$, Itô-expanding the function defined by (8) gives

$$\begin{aligned}\mathrm{d}F\big(t,S(t)\big) &= 1_{t\notin\{t_i\}\text{ or }S(t)>B(t)}F_S\big(t,S(t)\big)\,\sigma\big(t,S(t)\big)\,S(t)\,\mathrm{d}W(t)\\ &\quad+\sum_{i=1}^{n}\delta(t-t_i)\,1_{S(t_i)\le B(t_i)}\big[F(t_i+,S(t_i))-R(t_i)\big]\,\mathrm{d}t\end{aligned} \tag{9}$$

Integrating, taking expectations, and rearranging yields

$$F(0,S(0))=\mathrm{E}[g(S(T))]-\sum_{i=1}^{n}\mathrm{E}\Big[\big[F(t_i+,S(t_i))-R(t_i)\big]1_{S(t_i)\le B(t_i)}\Big]$$

The expectations can be substituted with integrals and European option spreads to give

$$F(0,S(0))=\mathrm{E}[g(S(T))]-\sum_{i=1}^{n}\int_0^{B(t_i)}[F(t_i+,S)-R(t_i)]\,C_{KK}(t_i,S)\,\mathrm{d}S \tag{10}$$

We have now arrived at an equation that explicitly specifies a static hedge for the barrier option. As for the continuous barrier case we need a position that replicates a European payoff $g(S)$ (for instance, the put–call portfolio (7)) and a number of butterfly spread positions along the barrier.[7] The number of spreads that we need is again dependent on the value of the barrier option along its barrier. As in (6), only barrier option values along the barriers depend directly on the model for stock price evolution.

2.3 A general result

The results in Sections 2.1 and 2.2 have been proved by Tanaka's formula. As one would expect, it is possible to prove the results by more traditional methods. Appendix A shows how this can be done through the use of differential forms and circulation theorems. The circulation theorems set out in Appendix A allow for a compact and completely general representation of barrier options with almost arbitrarily complicated knock-out regions. Such extensions can also be accomplished using the Tanaka formula. Specifically, we can summarize the results of the previous two subsections in the following theorem (where we arbitrarily have used European calls as the hedging instruments).

THEOREM 1 *Suppose that the underlying stock evolves according to (1) and consider an option that has the value $g(S(T))$ at time T and knocks out on a set $B \subset \Omega$, $\Omega = [0, T] \times (0, \infty)$, with a once-differentiable rebate function, R, that depends only on time. Assuming that $\Omega \backslash B$ is an open submanifold in Ω, a static hedge for the option value is defined by*

$$\begin{aligned} F(0, S(0)) = & \int_0^\infty g(S)\, C_{KK}(T, S)\, \mathrm{d}S \\ & - \frac{1}{2} \int_0^T \sum_{S \in (\overline{\partial B})(t, \cdot)} \left[F_S(t, S+) - F_S(t, S-) \right] \sigma(t, S)^2 S^2 C_{KK}(t, S)\, \mathrm{d}t \\ & - \int_0^\infty \sum_{t \in (|\partial B)(\cdot, S)} \left[F(t+, S) - F(t, S) \right] C_{KK}(t, S)\, \mathrm{d}S \\ & - \int_{\operatorname{int} B} R'(t)\, C_{KK}(t, S)\, \mathrm{d}t\, \mathrm{d}S \end{aligned}$$

where ∂B and $\operatorname{int} B$ denote, respectively, the boundary and interior of B, $R' = \mathrm{d}R/\mathrm{d}t$, and where we use the convention $F(T+, \cdot) = F(T, \cdot)$. Further, we define

$$\begin{aligned} |\partial B &= \left\{ (t, S) \in \partial B \,|\, \exists\, \varepsilon > 0 : (t, S + h) \in \partial B,\ \forall\, |h| < \varepsilon \right\} \\ \overline{\partial B} &= \partial B \setminus |\partial B \end{aligned}$$

and if $A \subset \Omega$, we let

$$\begin{aligned} A(t, \cdot) &= \left\{ S \in (0, \infty) \,|\, (t, S) \in A \right\} \\ A(\cdot, S) &= \left\{ t \in [0, T] \,|\, (t, S) \in A \right\} \end{aligned}$$

Although Theorem 1 looks complicated, it is really just a simple extension of the previous results. In particular, the barrier price is split into a contribution from the terminal maturity (first term), the non-vertical parts of the barrier (second term), the vertical parts of the barrier (third term), and the rebate (fourth term). Notice that the second term involves both $F_S(t, B+)$ and $F_S(t, B-)$; the former is required for down-and-out portions of the barrier, the latter for up-and-out parts of the barrier. Also, the technical requirement that $\Omega \backslash B$ is a submanifold is simply to ensure an "alive" region that is genuinely two-dimensional, ruling out arbitrarily crinkly or even fractal barriers.

As we have seen in the previous two sections, it is often possible to simplify the expressions above by either completing integrals[8] or applying the forward equation of Dupire (1994). However, care must be taken to ensure that the resulting expressions represent static hedges with no cashflows being generated on the alive region of the option.

We emphasize that Theorem 1 holds for diffusion processes of the type (1) but does not generalize to the case of stochastic volatility. Appendix B shows that, while a price representation similar to that in Theorem 1 is possible, the resulting expression does not represent a static hedge. For simple barrier options on martingale stock processes, stochastic volatility models can be accommodated by the static overhedges developed in Brown, Hobson and Rogers (1998). If it is known that volatility is restricted to a specified band (as in Avellaneda, Levy and Paras, 1995), it is possible to combine the Hamilton–Jakobi–Bellman equation with the approach taken in this paper to develop an overhedging strategy. Details of this are available from the authors on request.

2.4. Comparison with existing results

The hedge suggested by Theorem 1 generally involves taking on an infinite number of positions in European options with maturities in $[0, T]$, all struck at the barrier B. In contrast, the static hedges suggested by, for instance, Carr, Ellis and Gupta (1998) only involve taking positions in European options that mature at time T. Although the hedge they propose has so far only been proved to be possible for the fairly limited dynamics of the underlying (zero interest rates and dividends and a local volatility satisfying a certain symmetry condition), it is worth demonstrating that the two static hedges are, indeed, consistent. Further material on this can be found in Chou and Georgiev (1998).

Let us focus on the case of a down-and-out call option with constant barrier, strike $K > B$, and no rebate, for which the hedging relation (6) can be written as

$$F(0) = C(T, K) + \int_0^T F_{tS}(t, B+) P(t, B)\,\mathrm{d}t$$

For the case when volatility is constant and dividends and rates are zero, the integrand is given by

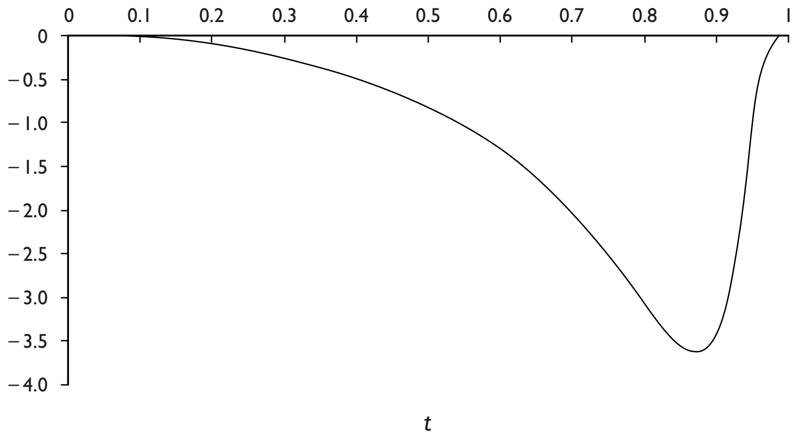
$$F_{tS}(t, B+) P(t, B) = \phi(x) \frac{\ln(B/K)}{\sigma(T-t)^{3/2}} \left(B\Phi(y + \sigma\sqrt{t}) - S\Phi(y) \right),$$

$$x = \frac{\ln(B/K) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$

$$y = \frac{\ln(S/B) - \frac{1}{2}\sigma^2 t}{\sigma\sqrt{t}} \tag{11}$$

(11) can be proven by taking the cross-derivative of the barrier option pricing expression in Merton (1973). In Figure 1 we give an example of the profile $\{F_{tS}(\cdot, B+) P(\cdot, B)\}$.

FIGURE 1 Static hedging portfolio for down-and-out call



The figure shows the profile $\{F_{tS}(\cdot, B+)P(\cdot, B)\}$ for the case of zero rates and dividends, $\sigma = 0.25$, and a down-and-out call option with $T = 1$, $B = 80$, $K = 100$, and $S(0) = 100$.

One can show that integrating expression (11) yields Merton's pricing formula:

$$F(0) = S\Phi(a) - K\Phi\left(a - \sigma\sqrt{T}\right) - \frac{K}{B}\left(\frac{B^2}{K}\Phi(b + \sigma\sqrt{T}) - S\Phi(b)\right),$$

$$a = \frac{\ln(S/K) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

$$b = \frac{\ln\left(S/(B^2/K)\right) - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

Carr, Ellis and Gupta (1998) observe that Merton's formula can be represented as

$$F(0) = C(T, K) - \frac{K}{B}P\left(T, B^2/K\right) \tag{12}$$

which leads these authors to suggest the static hedge:

- long one call with maturity T and strike K;
- short K/B puts with maturity T and strike B^2/K.

We point out again that the simpler representation (12) of (6) is possible only for very simple assumptions about the stock process.

3 Discontinuous asset dynamics

In this section we consider static hedging for the case where the process (1) is extended to allow the stock to jump. Specifically, we will assume that the stock evolves according to

$$\frac{\mathrm{d}S(t)}{S(t-)} = -\lambda(t)\,m(t)\,\mathrm{d}t + \sigma\big(t, S(t)\big)\,\mathrm{d}W(t) + \big(J(t)-1\big)\,\mathrm{d}N(t) \tag{13}$$

where N is a Poisson process with deterministic intensity $\lambda(t)$, and $\{J(t)\}_{t\geq 0}$ is a sequence of independent positive random variables, each with distribution given by the densities $\{\xi(t,\cdot)\}_{t\geq 0}$. We assume that W, N, and J are independent of each other, and let $m(t) = \mathrm{E}[J(t) - 1]$ denote the mean jump.

Let us now consider the case of a continuous down-and-out barrier option $F(t, S(t))$, equivalent to the discussion in Section 2.1. We will need the definition

$$\Delta F(t) = F\big(t, S(t)J(t)\big) - F\big(t, S(t)\big)$$

Itô–Tanaka expansion of F yields

$$\begin{aligned}\mathrm{d}F(t) = {} & 1_{S(t)>B(t)}\,\mathrm{d}M(t) + \tfrac{1}{2}\delta(S(t)-B(t))\,F_S(t, B(t)+)\,\sigma(t, S(t))^2 B(t)^2\,\mathrm{d}t \\ & + 1_{S(t)<B(t)}\big[R'(t)\,\mathrm{d}t + \Delta F(t)\,\mathrm{d}N(t)\big]\end{aligned}$$

where $M(t)$ is a (discontinuous) martingale. Integrating over time and taking expectations yields

$$\begin{aligned}\mathrm{E}\big[g(S(T))\big] - F(0) = {} & \frac{1}{2}\int_0^T F_S(t, B(t)+)\,\sigma(t, B(t))^2 B(t)^2 f(t, B(t))\,\mathrm{d}t \\ & + \int_0^T \mathrm{E}\Big[\big(R'(t) + \lambda(t)\,\Delta F(t)\big)1_{S(t)<B(t)}\Big]\,\mathrm{d}t\end{aligned}$$

Using (2), we get

$$\begin{aligned}F(0) = {} & \int_0^\infty g(S)\,P_{KK}(T, S)\,\mathrm{d}S \\ & - \frac{1}{2}\int_0^T \sigma(t, B(t))^2 B(t)^2 F_S(t, B(t)+)\,P_{KK}(t, B(t))\,\mathrm{d}t \\ & - \int_0^T\!\!\int_0^{B(t)} \big[R'(t) + \lambda(t)\,\mathrm{E}\big[\Delta F \,\big|\, S(t) = S\big]\big]\,P_{KK}(t, S)\,\mathrm{d}S\,\mathrm{d}t\end{aligned} \tag{14}$$

This shows that our static replication results can be extended to the case of jumps. In this case the static replicating portfolio also includes an extra term from below the barrier. To set up the static hedge, we need a model to compute F_S at the barrier, as well as the quantity $\mathrm{E}[\Delta F]$ below the barrier.

Andersen and Andreasen (1999a,b) show that, under the model assumptions above,

$$0 = -C_T + m(T)\,\lambda(T)\,KC_K + \frac{1}{2}\sigma(T,K)^2 K^2 C_{KK} + \lambda'(T)\,E'[\Delta' C]$$

$$0 = -P_T + m(T)\,\lambda(T)\,KP_K + \frac{1}{2}\sigma(T,K)^2 K^2 P_{KK} + \lambda'(T)\,E'[\Delta' P] \tag{15}$$

where $\lambda'(t) = \lambda(t)(1 + m(t))$ and

$$E'[\Delta' C](t,K) = \int_0^\infty \frac{J}{1+m(t)}\,\zeta(t,J)\,C(t,K/J)\,\mathrm{d}J - C(t,K)$$

$$E'[\Delta' P](t,K) = \int_0^\infty \frac{J}{1+m(t)}\,\zeta(t,J)\,P(t,K/J)\,\mathrm{d}J - P(t,K)$$

The quantities $\mathrm{E}'[\Delta' C]$ and $\mathrm{E}'[\Delta' P]$ can be interpreted as spreads on a continuum of European options around a certain strike. The fact that these spreads contain strikes that lie above the barrier means that we *cannot* generally use (15) to eliminate the term $\frac{1}{2}\sigma(t,B(t))^2 B(t)^2 C_{KK}$ in (14) without introducing cash-flows on the "alive" region of the barrier and thereby destroying the static hedge in (14). Nevertheless, if ζ is known, (15) does provide us with a way to compute the volatility function $\sigma(t,B(t))$ in (13) from quoted options prices; see Andersen and Andreasen (1999a,b).

We note that the hedging expression for discrete barriers (10) is unaffected by jumps. This together with (14) leads to the following generalization of Theorem 1.

THEOREM 2 *Suppose that the underlying stock evolves according to (11) and consider a barrier option similar to that in Theorem 1. A static hedge for the option is defined through*

$$\begin{aligned} F(0,S(0)) = {} & \int_0^\infty g(S)\,C_{KK}(T,S)\,\mathrm{d}S \\ & - \frac{1}{2}\int_0^T \sum_{S\in(\overline{\partial B})(t,\cdot)} \big[F_S(t,S+) - F_S(t,S-)\big]\,\sigma(t,S)^2 S^2 C_{KK}(t,S)\,\mathrm{d}t \\ & - \int_0^\infty \sum_{t\in(|\partial B)(\cdot,S)} \big[F(t+,S) - F(t,S)\big]\,C_{KK}(t,S)\,\mathrm{d}S \\ & - \int_{\mathrm{int}\,B} \big[R'(t) + \lambda(t)\,\mathrm{E}\big[\Delta F(t)\,\big|\,S(t) = S\big]\big]\,C_{KK}(t,S)\,\mathrm{d}t\,\mathrm{d}S \end{aligned}$$

where the notation is the same as in Theorem 1.

4 Some practical considerations

The results so far have relied on the key assumption that put and call options exist in unlimited supply and at all strikes and maturities. In practice, this assumption is obviously not satisfied, making the construction of perfect hedges impossible. In this section we will briefly deal with this issue, and we also consider the problem (which also affects regular dynamic hedging) that certain barrier option contracts have deltas at the barrier that grow infinitely large as the option approaches maturity.

4.1 Finite number of European options

In practical applications we only have a finite and often sparse set of actively traded options. This means that it can be difficult to put together a portfolio that closely replicates the barrier option under consideration. It is useful to consider the alternative of setting up static over- or underhedges. As a first example, consider the case of a down-and-out call with strike K, no rebate, and a discretely monitored, constant barrier B. The barrier observation dates are $\{t_i\}$. The hedging equation for this option contract is

$$F(0, S(0)) = C(T, K) - \sum_i \int_0^B F(t_i+, S)\, C_{KK}(t, S)\, \mathrm{d}S$$

It is clear that, to overhedge the option at time 0, we need to sell off a profile, $p_i(\cdot)$, that satisfies

$$\begin{aligned} p_i(S) &\le F(t_i, S), \qquad && S \le B \\ p_i(S) &\le 0, && S > B \end{aligned}$$

for each barrier observation date t_i. If for maturity t_i we can trade European call options with strikes $K_1^i, \dots, K_{m_i}^i$, we find that the cheapest overhedge of the option corresponds to the profile

$$p_i(s) = \sum_j a_j^i \left(S - K_j^i\right)^+$$

where the weights $\{a_j^i\}$ are the solution to the linear programming problem

$$\max_{\{a_j^i\}} \sum_j a_j^i C\left(t_i, K_j^i\right)$$

$$s.t. \ \sum_j a_j^i \left(S - K_j^i\right)^+ \le F(t_i, S) 1_{S \le B}, \quad \forall S$$

After discretizing in the stock price dimension the linear problem can be solved numerically using the simplex algorithm (see, eg, Press *et al.*, 1992).

Let us now turn to a slightly more complicated example where the option has a continuous down-and-out barrier at a constant level B. Assume that the rebate is constant over time but allow for a general payoff g. We assume that we can purchase enough T-maturity options at various strikes to allow for an overhedge of g. However, we can only transact in B-strike European put options with a finite number of maturities $0 = T_0, T_1, T_2, \ldots, T_{n-1}, T_n = T$. Assuming deterministic volatility of the underlying stock, the hedging equation is, from (6),

$$\begin{aligned} F(t, S(t)) &= \mathrm{E}_t[g(T)] - \int_t^T F_S(u, B+) P_T(t; u, B)\, \mathrm{d}u \\ &= \mathrm{E}_t[g(T)] - F_S(T, B+) P(t; T, B) + \int_t^T F_{St}(u, B+)\, P(t; u, B)\, \mathrm{d}u \\ &= \mathrm{E}_t[g(T)] - F_S(T, B+) P(t; T, B) + \sum_{i: T_i > t}^{n} \int_{t \wedge T_{i-1}}^{T_i} F_{St}(u, B+)\, P(t; u, B)\, \mathrm{d}u \end{aligned}$$

where the second equation follows from integration by parts and the fact that $P(t; t, B) = 0$ when $S(t) > B$. For our process assumption, European put and call option prices are increasing in maturity, whereby we can now write

$$\underline{V}(t) \le F\big(t, S(t)\big) \le \overline{V}(t) \tag{16}$$

where

$$\begin{aligned} \underline{V}(t) = {} & \mathrm{E}_t[g(T)] - F_S(T, B+) P(t; T, B) \\ & + \sum_{i: T_i > t}^{n} \big[F_S(T_i, B+) - F_S(t \wedge T_{i-1}, B+) \big]\, P(t; \underline{T}_i, B) \end{aligned} \tag{17a}$$

$$\begin{aligned} \overline{V}(t) = {} & \mathrm{E}_t[g(T)] - F_S(T, B+) P(t; T, B) \\ & + \sum_{i: T_i > t}^{n} \big[F_S(T_i, B+) - F_S(t \wedge T_{i-1}, B+) \big]\, P(t; \overline{T}_i, B) \end{aligned} \tag{17b}$$

$$\underline{T}_i = \begin{cases} T_i, & F_S(T_i, B+) \le F_S(t \wedge T_{i-1}, B+) \\ t \wedge T_{i-1}, & F_S(T_i, B+) > F_S(t \wedge T_{i-1}, B+) \end{cases}$$

$$\overline{T}_i = \begin{cases} t \wedge T_{i-1}, & F_S(T_i, B+) \le F_S(t \wedge T_{i-1}, B+) \\ T_i, & F_S(T_i, B+) > F_S(t \wedge T_{i-1}, B+) \end{cases}$$

To test the tightness of the above bounds on F, consider the special case of a down-and-out call option with strike $K = 100$, maturity $T = 1$, spot asset price

TABLE 1 Under- and overhedging of down-and-out call

n	Underhedge $\underline{V}(0)$	Mid-sum	Overhedge $\overline{V}(0)$
∞	7.1791	7.1791	7.1791
365	7.1743	7.1790	7.1838
183	7.1695	7.1790	7.1885
122	7.1648	7.1791	7.1933
91	7.1601	7.1791	7.1980
73	7.1553	7.1791	7.2028
52	7.1458	7.1791	7.2123
37	7.1317	7.1820	7.2266
18	7.0850	7.1826	7.2747
12	7.0395	7.1846	7.3241
8	6.9749	7.1912	7.4019
6	6.9151	7.2028	7.4850
4	6.8086	7.2404	7.6665
3	6.7168	7.2948	7.8672
2	6.5660	7.4456	8.3195

The table shows under- and over-hedges for a one-year, continuously monitored, down-and-out call option with strike $K = 100$ and barrier $B = 90$. The spot price is $S(0) = 100$ and the rebate amount is 0. Over- and under-hedge prices are computed from (17a,b) and are reported as a function of n, the number of maturities at which puts struck at B can be purchased in the market. The maturities of the replicating put portfolio are assumed to be equidistantly spaced on $[0, T]$.

$S(0) = 100$, and barrier $B = 90$. We assume that the stock volatility is constant at $\sigma = 0.25$. For this case, a closed-form solution exists for the option (see Section 2.4), and all terms in (16) can be computed without resolving to numerical methods. Notice also that here $F_S(T, B+)P(t; T, B) = 0$, $\mathrm{E}_t[g(T)] = C(t; T, K)$, and $F_{S_t} \leq 0$ at the barrier whereby $\overline{T}_i = t \wedge T_{i-1}$ and $\underline{T}_i = T_i$. Table 1 shows the bounds in (16) as a function of the number of equally spaced put maturities, n. For reference we also report the value of a hedge based on a simple mid-sum approximation to the integral (that is, we simply substitute $P(t; T_i, B)$ for $P(t; \underline{T}_i, B)$ in (17a)).

4.2 Unbounded delta

For many barrier options, the terminal payoff function g is discontinuous at the barrier, resulting in an unbounded delta at the maturity of the barrier option. Common examples include continuously monitored down-and-out puts with strike above the barrier, and continuously monitored up-and-out call options with strike below the barrier. The unbounded delta of such "in-the-money" barrier options is not only a problem for traditional dynamic delta hedging but also affects our static hedges, which involve option spread positions of size proportional to the delta at the barrier.

Let us focus on the specific example of an up-and-out call option with a flat, continuous barrier B, no rebate, and a strike $K < B$. Our hedging equation is

$$F(0) = C(T, K) - C(T, B) + (B - K)\, C_K(T, B) + \int_0^T F_S(t, B-)\, C_T(t, B)\, \mathrm{d}t$$

Here we use calls in the replicating portfolio to avoid cashflows before the option expires or knocks out. Since $F_S(t, B-) \to -\infty$ for $t \uparrow T$, we would need to short an infinite number of maturity spreads in the hedge portfolio. To circumvent this problem we note that, if we move the barrier slightly upwards by $\varepsilon > 0$ without changing the terminal pay-off,[9] we not only get an overhedge but also are able to bound the delta at the barrier. In fact, the delta of this option at $S = B$ will tend to zero as we approach maturity. The resulting overhedge is

$$\overline{F}(0) = C(T, K) - C(T, B) + (B - K)\, C_K(T, B) + \int_0^T \overline{F}_S(t, \varepsilon + B-)\, C_T(t, B + \varepsilon)\, \mathrm{d}t$$

The choice of ε is a matter of compromise: the larger ε is, the more expensive the hedge becomes; the smaller ε is, the larger (in absolute magnitude) the delta can become.

A more scientific approach to the problem of unbounded deltas has been suggested by Wystup (1997), and Schmock, Shreve and Wystup (1999). The authors impose constraints on the delta and show that the cheapest super-replicating claim that satisfies this constraint can be found as the solution to a stochastic control problem. Interestingly, Wystup (1997) points out that the simple strategy of moving the barrier is a close approximation of the "correct" strategy. He also gives an approximate link between the size of the barrier shift (ε above) and the constraint on delta.

5 Conclusion

This paper has discussed the construction of static hedges for generalized barrier-type claims on stocks following a jump–diffusion process with state- and time-dependent volatility. The static hedge takes the form of a linear portfolio of European puts and calls that exactly matches the cashflow from the option to be hedged. Allowing for time-dependent rebates, we have derived exact expressions for the composition of the hedging portfolio, the form of which depends both on the option to be hedged and on the stock process. While our theoretical results assume an unlimited supply of European options and perfect knowledge of stock dynamics, we have discussed several practical techniques for relaxing such idealized assumptions.

Finally, we point out that, although this paper has focused on barrier options, many other option types allow for a decomposition in terms of barrier options which again allows our hedging results to be applied. For instance, lookback and "ratchet" options can be synthesized by a "ladder" of continuously monitored barrier options (see, eg, Carr and Chou 1997), and can thus be statically hedged in our framework. Similarly, Bermudan options can, after the determination of the early exercise frontier, be treated as a discretely monitored barrier option (albeit with an asset-dependent rebate) and can be hedged by a static position of puts and calls that mature at each exercise date.

Appendix A: Derivation of hedging equation using differential forms

Let $f(t,S)$ denote the density of S that satisfies (1), and let $F(t,S)$ be the value of a knock-out option that knocks out on some set $B \subset \Omega$, $\Omega = [0,T] \times (0,\infty)$, where B is closed in Ω. Let $B^c = \Omega \backslash B$ denote the complement of B, and define the open set $\hat{B} = B^c \backslash \{(t,S): t = 0 \text{ or } t = T\}$. We assume that $\hat{B}$ is a submanifold of Ω. Consider the differential form

$$\omega = Q(t,S)\,\mathrm{d}S + P(t,S)\,\mathrm{d}t,$$

$$Q(t,S) = -f(t,S)F(t,S)$$

$$P(t,S) = \frac{1}{2}\frac{\partial F(t,S)}{\partial S}\sigma(t,S)^2 S^2 f(t,S) - \frac{1}{2}F\frac{\partial\left(\sigma(t,S)^2 S^2 f(t,S)\right)}{\partial S}$$

LEMMA A1 *Let the submanifold $\hat{B}$ be as defined above, and let there be given a submanifold $M \subset \hat{B}$, with boundary curve ∂M lying entirely in $\hat{B}$. Then*

$$\int_{\partial M} \omega = 0$$

PROOF Given the assumptions about the topology of M, proving Lemma A1 is equivalent to showing that ω is closed in M, ie, that

$$\frac{\partial Q}{\partial t} = \frac{\partial P}{\partial S}$$

for all $(t,S) \in M$. Now

$$-\frac{\partial Q}{\partial t} + \frac{\partial P}{\partial S} = \left[f\frac{\partial F}{\partial t} + F\frac{\partial f}{\partial t}\right] + \left[\tfrac{1}{2}\sigma^2 S^2 f\frac{\partial^2 F}{\partial S^2} - \tfrac{1}{2}F\frac{\partial^2(\sigma^2 S^2 f)}{\partial S^2}\right]$$

$$= f\left[\frac{\partial F}{\partial t} + \tfrac{1}{2}\sigma^2 S^2\frac{\partial^2 F}{\partial S^2}\right] + F\left[\frac{\partial f}{\partial t} - \tfrac{1}{2}\frac{\partial^2(\sigma^2 S^2 f)}{\partial S^2}\right]$$

On the submanifold M, F satisfies the backward equation (2), whereby the term multiplying f is zero. By the Fokker–Planck equation, we also have, for $t > 0$,

$$\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2(\sigma^2 S^2 f)}{\partial S^2} = 0, \quad \text{subject to } f(0,S) = \delta(S - S(0))$$

whence the term on F is also zero. □

As an application of Lemma A1, consider now the case of a down-and-out barrier option where $B = \{(t,S): S \le B(t),\ t \in [0,T]\}$, for some continuous, positive function $B(t)$. Set

$$M = \{(t,S): S \in [B(t)+\varepsilon, L],\ t \in [\varepsilon, T-\varepsilon]\}$$

for two parameters $\varepsilon > 0$ and L, where everywhere $L > B(t) + \varepsilon$. Integrating around the boundary of M, and letting $L \to \infty$ and $\varepsilon \downarrow 0$, we get from the lemma:

$$\begin{aligned} F(0,S(0)) &= \\ &\int_{B(T)}^{\infty} f(T,S)F(T,S)\,\mathrm{d}S - \frac{1}{2}\int_0^T F_S(t,B(t)+)\,\sigma(t,B(t)+)^2\,(B(t))^2 f(t,B(t))\,\mathrm{d}t \\ &+ \frac{1}{2}\int_0^T R(t)\left.\frac{\partial\left(\sigma(t,S)^2 S^2 f(t,S)\right)}{\partial S}\right|_{S=B(t)+}\mathrm{d}t + \int_0^T f(t,B(t))R(t)B'(t)\,\mathrm{d}t \end{aligned} \tag{A1}$$

where we have used that

$$\lim_{\varepsilon\downarrow 0}\int_{B+\varepsilon}^{L} f(0,S)F(0,S)\,\mathrm{d}S = F(0,S(0))$$

and assumed that $f(t,S)$ dies out sufficiently fast when S is increased to make the integral along $S = L$ vanish in the limit. In (A1) we have introduced the rebate $R(t) = F(t,B(t))$.

To complete the derivation, integration of the Fokker–Planck equation yields

$$\begin{aligned} &\frac{1}{2}\frac{\partial}{\partial S}\left[\sigma(t,S)^2 S^2 f(t,S)\right]_{S=B(t)} = \\ &\int_0^{B(t)}\frac{\partial f(t,S)}{\partial t}\,\mathrm{d}S = -f(t,B(t))B'(t) + \frac{\partial}{\partial t}\int_0^{B(t)} f(t,S)\,\mathrm{d}S \end{aligned}$$

Inserting this into (A1) and performing integration by parts yields the desired result:

$$\begin{aligned} F(0,S(0)) = \mathrm{E}\left[F(T,S)\right] &- \frac{1}{2}\int_0^T F_S(t,B(t)+)\,\sigma(t,B(t)+)^2 B(t)^2 f(t,B(t))\,\mathrm{d}t \\ &- \int_0^T \mathrm{E}\left[1_{S(t)\le B(t)}\right] R'(t)\,\mathrm{d}t \end{aligned}$$

While Lemma A1 is completely general and can be applied to almost all types of barrier options, it is slightly inconvenient to work with and requires some rearrangements of the final results to yield a static hedge. Below, we have listed a more convenient form of Lemma A1 expressed directly in terms involving puts, $P(T, K)$:[10]

LEMMA A2 *Let everything be as in Lemma A1, and define*

$$\begin{aligned}\bar{\omega} &= \left[\frac{1}{2}F_S(t,S)\,\sigma(t,S)^2 S^2 P_{KK}(t,S) + P_K(t,S)\,F_t(t,S)\right]\mathrm{d}t + P_K(t,S)\,F_S(t,S)\,\mathrm{d}S \\ &= \frac{1}{2}F_S(t,S)\,\sigma(t,S)^2 S^2 P_{KK}(t,S)\,\mathrm{d}t + P_K(t,S)\,\mathrm{d}F\end{aligned}$$

Then

$$\int_{\partial M} \bar{\omega} = 0$$

PROOF Set $Z(t,S) = F(t,S)\int_0^S f(t,s)\,\mathrm{d}s$ and notice that

$$\begin{aligned}\mathrm{d}Z(t,S) = {} & F_t(t,S)\int_0^S f(t,s)\,\mathrm{d}s\,\mathrm{d}t + F(t,S)\int_0^S f_t(t,s)\,\mathrm{d}s\,\mathrm{d}t \\ & + F_S(t,S)\int_0^S f(t,s)\,\mathrm{d}s\,\mathrm{d}S + F(t,S)f(t,S)\,\mathrm{d}S \end{aligned} \tag{A2}$$

From Lemma 1, on M,

$$\begin{aligned}\omega = {} & \frac{1}{2}F_S(t,S)\,\sigma(t,S)^2 S^2 f(t,S)\,\mathrm{d}t \\ & - \frac{1}{2}F(t,S)\,\frac{\partial\left(\sigma(t,S)^2 S^2 f(t,S)\right)}{\partial S}\,\mathrm{d}t - f(t,S)\,F(t,S)\,\mathrm{d}S \\ = {} & \frac{1}{2}F_S(t,S)\,\sigma(t,S)^2 S^2 f(t,S)\,\mathrm{d}t - F(t,S)\int_0^S f_t(t,s)\,\mathrm{d}s\,\mathrm{d}t - f(t,S)\,F(t,S)\,\mathrm{d}S \\ = {} & \frac{1}{2}F_S(t,S)\,\sigma(t,S)^2 S^2 f(t,S)\,\mathrm{d}t - \mathrm{d}Z(t,S) + \mathrm{d}F(t,S)\int_0^S f(t,s)\,\mathrm{d}s\end{aligned}$$

Here the first equation follows from the Fokker–Planck equation, and the second from (A2). As $\mathrm{d}Z$ is an exact differential, the lemma follows by application of (2). □

As a simple example, consider applying Lemma A2 to a down-and-out option with a single step-down discontinuity at $t = T^*$. Specifically, we set

$$B(t) = \begin{cases} B_1(t), & 0 \le t \le T^* \\ B_2(t), & T^* < t \le T \end{cases}$$

where B_1 and B_2 are smooth functions, with $B_1(T^*) > B_2(T^*+)$. Using the same type of integration contour as in our previous example, we now get:

1. Time 0+ vertical piece:

$$\int_{\infty}^{B(0)+} P_K(0, S)\, F_S(0, S)\, \mathrm{d}S = -F(0, \infty) + F(0, S(0))$$

2. Piece along $B_1(t)+$, for $t \in (0, T^*)$ (where $\mathrm{d}F = R'(t)\,\mathrm{d}t$):

$$\frac{1}{2}\int_0^{T^*-} F_S\big(t, B_1(t)+\big)\,\sigma\big(t, B_1(t)\big)^2 B_1(t)^2 P_{KK}\big(t, B_1(t)\big)\,\mathrm{d}t$$

$$+\int_0^{T^*-} P_K\big(t, B_1(t)\big)\, R'(t)\,\mathrm{d}t$$

3. Horizontal piece from $(t, S) = (T^*-, B_1(T^*)+)$ to $(t, S) = (T^*+, B_1(T^*)+)$:

$$P_K\big(T^*, B_1(T^*)\big)\Big(F\big(T^*+, B_1(T^*)\big) - R(T^*)\Big)$$

4. Time T^*+ vertical piece:

$$\int_{B_1(T^*)}^{B_2(T^*+)} P_K(T^*, S)\, F_S(T^*+, S)\,\mathrm{d}S =$$

$$P_K\big(T^*, B_2(T^*+)\big)\, R(T^*) - P_K\big(T^*, B_1(T^*)\big)\, F\big(T^*+, B_1(T^*)\big)$$

$$-\int_{B_1(T^*)}^{B_2(T^*+)} F(T^*+, S)\, P_{KK}(T^*, S)\,\mathrm{d}S$$

5. Piece along $B_2(t)+$, for $t \in (T^*, T)$:

$$\frac{1}{2}\int_{T^*+}^{T} F_S(t, B_2(t)+)\,\sigma(t, B_2(t))^2 B_2(t)^2 P_{KK}(t, B_2(t))\,\mathrm{d}t$$

$$+\int_{T^*}^{T} P_K(t, B_2(t))\, R'(t)\,\mathrm{d}t$$

6. Time T– vertical piece:

$$\int_{B_2(T)+}^{\infty} P_K(T, S)\, F_S(T, S)\, \mathrm{d}S$$

$$= F(T, \infty) - P_K(T, B_2(T))\, R(T) - \int_{B_2(T)+}^{\infty} g(S)\, P_{KK}(T, S)\, \mathrm{d}S$$

$$= F(T, \infty) - \int_0^{\infty} g(S)\, P_{KK}(T, S)\, \mathrm{d}S$$

7. Horizontal piece at $S = L$, $L \to \infty$:

$$\int_T^0 P_K(t, \infty)\, F_t(t, \infty)\, \mathrm{d}t = F(0, \infty) - F(T, \infty)$$

Adding all pieces, setting the sum to zero, and rearranging yields the desired static hedge decomposition:

$$F(0, S(0)) = \int_0^{\infty} g(S)\, P_{KK}(T, S)\, \mathrm{d}S$$

$$- \frac{1}{2}\int_0^T F_S\big(t, B(t)+\big)\sigma\big(t, B(t)\big)^2 B(t)^2 P_{KK}\big(t, B(t)\big)\, \mathrm{d}t$$

$$- \int_0^T P_K\big(t, B(t)\big) R'(t)\, \mathrm{d}t - \int_{B_1(T^*)}^{B_2(T^*+)} \big(F(T^*+, S) - R(T^*)\big) P_{KK}(T^*, S)\, \mathrm{d}S$$

Finally, we wish to demonstrate that it is possible to formulate Theorem 1 in terms of circulation integrals. Consider the following:

THEOREM A1 *Let everything be as in Lemmas A1 and A2. Let the connected components of the knock-out set B be denoted as B_i. Then*

$$F(0, S(0)) = \mathrm{E}\big[F(S(T))\big] - \sum_i \int_{\partial B_i +} \overline{\omega} \tag{A3}$$

where $\partial A+$ for a set A indicates a contour infinitesimally close to ∂A but just outside the set A wherever $\partial A \subset \operatorname{int}\Omega$, and which coincides with ∂A otherwise. The circulation integral in (A3) should be performed counterclockwise.

PROOF We define ω and $\overline{\omega}$ as in Lemmas A1 and A2, but, using the rebate function $R(t)$, we extend their domains of definition from $\hat{B}$ to all of $\overline{\Omega}$, the closure of Ω in $\Re^2$. $\overline{\Omega}$ is a compact space, whose boundary includes the points at $S = \infty$. (We may alternatively obtain this type of boundary by a standard limiting procedure, as demonstrated earlier). The forms so defined contain singularities (at ∂B, and at $t = 0$ and $t = T$); however, these are all integrable singularities as their component functions are products of derivatives of piecewise-smooth functions, bounded on compact subsets. These forms are closed everywhere in B^c, and so we may choose a contour $\partial B^c - \subset B^c$ that is infinitesimally close to ∂B^c. By Lemmas A1 and A2 we find that

$$\int_{\partial B^c -} \omega = \int_{\partial B^c -} \bar{\omega} = 0$$

Now, B^c satisfies $B^c = \bar{\Omega}\backslash\bar{B}$, so that $\partial B^c = \partial\bar{\Omega}\backslash\partial\bar{B}$ in $\Re^2$, and also, that $\partial B^c - = \partial\bar{\Omega}\backslash\partial\bar{B}+$, so that

$$\int_{\partial B^c -} \omega = \int_{\partial\bar{\Omega}} \omega - \sum_i \int_{\partial B_i +} \omega = 0; \qquad \int_{\partial B^c -} \bar{\omega} = \int_{\partial\bar{\Omega}} \bar{\omega} - \sum_i \int_{\partial B_i +} \bar{\omega} = 0$$

Note that the integrals on the right may run over much larger regions than those on the left, but these additional integrals cancel out. The extra pieces are in $\partial\bar{\Omega} \cap \partial\bar{B}$ (closure in $\Re^2$), and represent integrals along the lines $\{S = 0, t \in [0, T]\}$ and $\{S = \infty, t \in [0, T]\}$ or the other parts of $\partial\bar{\Omega}$. These extra pieces make use of the extension of the forms ω and $\bar{\omega}$ to $\bar{\Omega}$, because they are not in ∂B^c, and so ω and $\bar{\omega}$ on these contours cannot be obtained as a limit of values in B^c.

Finally, we note that $\omega = \bar{\omega} + \mathrm{d}Z$, with Z defined in the proof of Lemma A2. The function Z is well defined everywhere in $\bar{\Omega}$, and, as $\mathrm{d}Z$ is an exact form on all of $\bar{\Omega}$,

$$\int_{\partial\bar{\Omega}} \bar{\omega} = \int_{\partial\bar{\Omega}} \omega$$

Using the same technique as used in the example after Lemma A1, it is easy to verify that, integrating counterclockwise,

$$\int_{\partial\bar{\Omega}} \bar{\omega} = -F(0, S(0)) + \mathrm{E}\big[F(T, S(T))\big]$$

Thus, we have the final result. □

Appendix B: Stochastic volatility

Consider now the case where S follows the process

$$\frac{\mathrm{d}S(t)}{S(t)} = \sigma(t)\,\mathrm{d}W(t)$$

where $\sigma(t)$ is a stochastic process. As in Section 2.1, let F denote the price of a down-and-out option with a continuous barrier. Since the volatility is allowed to be stochastic, F will in general depend on other variables than time and stock price level, ie,

$$F(t) = F\big(t, S(t), x(t)\big)$$

where x is a vector of state variables additional to time and current stock price. So Itô–Tanaka expansion of F yields

$$\begin{aligned}\mathrm{d}F(t) &= 1_{S>B}\,\mathrm{d}M(t) + 1_{S<B}\,R'(t)\,\mathrm{d}t \\ &+ \frac{1}{2}\,\delta\big(S(t)-B(t)\big)F_S\big(t, B(t)+, x(t)\big)\,\sigma(t)^2 B(t)^2\,\mathrm{d}t \\ &+ 1_{S=B}\left[\sum_i F_{x_i S}\,\mathrm{d}x_i\,\mathrm{d}S + \frac{1}{2}\sum_i\sum_j F_{x_i x_j}\,\mathrm{d}x_i\,\mathrm{d}x_j\right]\end{aligned}$$

where M is a martingale. However, if the limit F_{x_i} for $S \downarrow B$ exists and is finite almost everywhere, then continuity on $\{S > B\}$ and the fact that $F\big|_{S=B} = R$ imply that $F_{x_i} \to 0$ and hence we can ignore the terms in the sums. So, integrating over time and taking expectations yields

$$\begin{aligned}F(t) &= \mathrm{E}_t\big[g(S(T))\big] - \int_t^T R'(u)\,\mathrm{E}_t\big[1_{S(u)<B(u)}\big]\,\mathrm{d}u \\ &\quad - \frac{1}{2}\int_t^T \mathrm{E}_t\big[F_S(u, B(u)+, x(u))\,\sigma^2(u)\,\big|\,S(u) = B(u)\big]\,B(u)^2\,C_{KK}\big(t; u, S(u)\big)\,\mathrm{d}u\end{aligned} \tag{B1}$$

Although equation (B1) is a perfectly valid expression for the price of the barrier option, it does not constitute a static hedge. The reason is, of course, that the terms $\mathrm{E}_t[F_S(u, B(u)+, x(u))\sigma^2(u)\,|\,S(u) = B(u)]$ are stochastic and move around as calendar time passes. As a consequence, any butterfly hedge set up to replicate the last integral in (B1) would need rebalancing over time. We note that (B1) may in some circumstances lead to static *over-* and *underhedges*, if one can find a robust way to bound $\mathrm{E}_t[F_S(u, B(u)+, x(u))\sigma^2(u)\,|\,S(u) = B(u)]$.

If the barrier is discretely monitored, as in Section 2.2, we find the expression

$$\begin{aligned}F(t) &= \mathrm{E}_t\big[g(S(T))\big] \\ &\quad - \sum_{i:\,t_i \geq t}\int_t^{B(t_i)}\Big(\mathrm{E}_t\big[F\big(t_i+, S, x(t_i)\big)\big] - R(t_i)\Big)\,C_{KK}\big(t; t_i, S(t_i)\big)\,\mathrm{d}S\end{aligned} \tag{B2}$$

Again, this expression does not represent a static hedge.

1. Notice that, if rates and dividend yields are non-zero but deterministic, one can easily represent the evolution of the underlying as in (1) by simply modeling the forward stock price. In this case barrier levels must be represented in terms of forward stock levels and terminal payments and rebates in terms of their discounted values. As our approach is valid for arbitrary barrier shapes (not just constant barriers), such transformations can easily be accommodated in the framework of this paper.

2. This assumption is made mainly for convenience. In most cases it is possible to allow for rebate functions with kinks and even discontinuities by interpreting derivatives of R in terms of step and delta functions.

3. Appendix B takes a closer look at stochastic volatility models.

4. For a relation between F_S and passage times in the Dupire forward PDE, see Chou and Georgiev (1998).

5. It is obvious from (2) that a decomposition using call spreads is possible, too. However, this would not constitute a static hedge as the call positions would generate random cashflows in the "alive" region of the barrier option.

6. For up-style barriers, a static hedge representation using calendar spreads (such as (6)) must be based on calls rather than puts to prevent the hedge from generating cashflows before the barrier options matures or knocks out. Such considerations are not necessary for the representation (5), which can be based on either puts or calls.

7. Obviously, we can use the same trick that leads to (7) to rewrite the position in butterfly spreads to a more "direct" position in put and call options.

8. Specifically, as R is a function only of time, it is clear that we can write the plane integral over the interior of B (last integral in the theorem) as a path integral over the boundary of B.

9. That is, we keep $g(S) = (S - K)^+ 1_{S<B}$.

10. By (2), Lemma A2 can also easily be written in terms of call options.

REFERENCES

Andersen, L., and Andreasen, J. (1999a). Jump–diffusion processes: volatility smile fitting and numerical methods for pricing. Forthcoming in *Review of Derivatives Research*.

Andersen, L., and Andreasen, J. (1999b). Jumping smiles. *Risk*, November, 65–8.

Andersen, L., Andreasen, J., and Brotherton-Ratcliffe, R. (1998). The passport option. *Journal of Computational Finance* **1** (3), 15–37.

Andersen, L., and Brotherton-Ratcliffe, R. (1998). The equity option volatility smile: a finite difference approach. *Journal of Computational Finance* **1** (2), 5–38.

Avellaneda, M., Levy, A., and Paras, A. (1995). Pricing and hedging derivative securities in markets with uncertain volatility. *Applied Mathematical Finance* **2,** 73–88.

Black, F., and Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy* **81,** 637–54.

Bowie, J., and Carr, P. (1994). Static simplicity. *Risk*, August, 45–9.

Brown, H., Hobson, D., and Rogers, L. (1998). Robust hedging of barrier options. Working paper, University of Bath.

Carr, P., and Chou, A. (1997). Hedging complex barrier options. Working Paper, Morgan Stanley and MIT Computer Science.

Carr, P., Ellis, K., and Gupta, V. (1998). Static hedging of exotic options. *Journal of Finance* **53,** 1165–91.

Chou, A., and Georgiev, G. (1998). A uniform approach to static hedging. *Journal of Risk* **1**(1).

Derman, E., Ergener, D., and Kani, I. (1995). Static options replication. *Journal of Derivatives* **2**(4), 78–95.

Dupire, B. (1994). Pricing with a smile. *Risk*, January, 18–20.

Karatzas, I., and Shreve, S. (1991). *Brownian Motion and Stochastic Calculus.* Springer.

Merton, R. (1973). Theory of rational option pricing. *Bell Journal of Economics* **4,** 141–84.

Press, W., Teukolsky, S., Vetterling, W., and Flannery, B. (1992). *Numerical Recipes in C*. Cambridge University Press.

Schmock, W., Shreve, S., and Wystup, U. (1999). Valuation of exotic options under shortselling constraints. Working paper, Carnegie Mellon University.

Wystup, U. (1997). Barrier options: a hedging problem and a hedging solution. Working paper, Carnegie Mellon University.