Imperial College London

Course: M5MF6
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MSc EXAMINATIONS IN MATHEMATICS AND FINANCE
DEPARTMENT OF MATHEMATICS

April 2016

M5MF6

Advanced Methods in Derivatives Pricing

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Advanced Methods in Derivatives Pricing

Date: April 2016 Time:

Answer all questions.

The total number of points is 100, and the precise grading is indicated in the text.

The rigour and clarity of your answers will be taken into account in the final grade.

Each problem is independent of the others.

1 [15 points] Warm-up: Preliminary questions

(i) [2 points] State the definition of an $(\mathcal{F}_t)_{t\geq 0}$-stopping time.

(ii) [2 points] State the definition of a $(\mathcal{F}_t)_{t\geq 0}$-adapted strict local martingale.

(iii) [2 points] Let $\xi : [0,\infty) \to (0,\infty)$ be a smooth bounded function, and let S be the unique strong solution to the stochastic differential equation $\mathrm{d}S_t = \xi(t)S_t\mathrm{d}W_t$ starting from $S_0 = 1$, where W is a standard Brownian motion. Assuming that the market has zero interest rates and that dividends are null, compute the implied volatility in this model, for each maturity $T \geq 0$ and strike $K \geq 0$.

(iv) [3 points] Define and explain in no more than five lines what the local volatility is.

(v) [6 points] Let S denote a strictly positive strict local martingale, and assume no interest rate nor dividend. Fix $T > 0$ and consider the two functions $f, g : [0,\infty) \to \mathbb{R}$ defined by:

$$\begin{aligned} f :& K \mapsto \mathbb{E}\left(S_T - K\right)_+, \\ g :& K \mapsto \mathbb{E}\left(K - S_T\right)_+ + S_0 - K. \end{aligned}$$

Explain any arbitrage issues using either (or both) function as a definition of a European Call option at inception. What are the consequences on the implied volatility?

2 [25 points] Tail asymptotics of the implied volatility

We shall investigate here the small- and large-strike behaviours of the implied volatility for a Compound Poisson process, using Roger Lee's Moment formula. Let S be a true non-negative martingale on $(\Omega, \mathcal{F}, \mathbb{P})$. We let $x := \log(K/S_0)$, for $K > 0$, denote the log-moneyness and we assume interest rates are null. Furthermore, for any $(x,t) \in \mathbb{R} \times (0,\infty)$, $\sigma_t(x)$ shall represent the implied volatility corresponding to European option prices with maturity t and log-moneyness x. Define the function $\psi : \mathbb{R}_+ \to \mathbb{R}$ by

$$\psi(z) \equiv 2 - 4\left(\sqrt{z(1+z)} - z\right). \tag{1}$$

Theorem 2.1 (Roger Lee's Moment Formula)**.**

- *[**Right Wing**] Let* $p^* := \sup\{p \geq 0 : \mathbb{E}(S_t^{1+p}) < \infty\}$ *and* $\beta_R := \limsup_{x\uparrow+\infty} \frac{\sigma_t^2(x)t}{x}$. *Then*

$$p^* = \frac{1}{2\beta_R} + \frac{\beta_R}{8} - \frac{1}{2}, \qquad \textit{or equivalently} \qquad \beta_R = \psi(p^*).$$

- *[**Left Wing**] let* $q^* := \sup\{q \geq 0 : \mathbb{E}(S_t^{-q}) < \infty\}$ *and* $\beta_L := \limsup_{x\downarrow-\infty} \frac{\sigma_t^2(x)t}{|x|}$. *Then*

$$q^* = \frac{1}{2\beta_L} + \frac{\beta_L}{8} - \frac{1}{2}, \qquad \textit{or equivalently} \qquad \beta_L = \psi(q^*).$$

(i) [3 points] Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. State the definition of $\limsup_{n\uparrow+\infty} u_n$ and $\liminf_{n\uparrow+\infty} u_n$, and give an example of a sequence for which the two limits do not coincide.

(ii) [3 points] Study the smoothness of the function ψ in (1), and compute its limits at zero and infinity.

(iii) [4 points] Consider the Black-Scholes model, in which the stock price process is the unique strong solution to $\mathrm{d}S_t = \xi S_t \mathrm{d}W_t$, with $S_0 = 1$, $\xi > 0$, and where W is a standard Brownian motion. Compute p^*, β_R, q^* and β_L from Theorem 2.1. What can you conclude?

(iv) We now consider a more sophisticated model for the stock price:

$$S_t = \exp\left(\gamma t + \sigma W_t + \sum_{n=1}^{N_t} Y_n\right), \qquad \text{for } t \geq 0,$$

where W is a Brownian motion, $\sigma > 0$, $\gamma \in \mathbb{R}$; here $(N_t)_{\geq 0}$ is a Poisson process with rate $\lambda > 0$, and the $(Y_n)_n$ forms a family of independent random variables with common distribution

$$\mathbb{P}(Y_1 \in \mathrm{d}x) = \mathrm{p}\lambda_+ \exp\Big(-\lambda_+ x\Big)\mathbb{1}_{\{x>0\}}\mathrm{d}x + (1-\mathrm{p})\lambda_- \exp\Big(-\lambda_-|x|\Big)\mathbb{1}_{\{x<0\}}\mathrm{d}x,$$

with $\mathrm{p} \in [0,1]$ and $\lambda_-, \lambda_+ > 0$, so that is S experiences both positive and negative jumps. We assume that both N and the family (Y_n) are independent of the driving Brownian motion W.

(a) Prove that

$$\mathbb{E}\left(\mathrm{e}^{uY_1}\right) = \mathrm{p}\frac{\lambda_+}{\lambda_+ - u} + (1-\mathrm{p})\frac{\lambda_-}{\lambda_- + u}, \qquad \text{for all } u \in \mathcal{D}_Y,$$

where the effective domain $\mathcal{D}_Y$ should be made explicit.

(b) [6 points] For any $t \geq 0$, compute $\mathbb{E}\left(S_t^u\right)$, for any u in some domain to determine.

(c) [2 points] Determine the value of γ ensuring that the process $(S_t)_{t\geq 0}$ is a true martingale.

(d) [2 points] Deduce p^* and q^* as given in Roger Lee's Moment Formula (Theorem 2.1).

(e) [2 points] How do the wings of the implied volatility evolve with maturity? Quote a model (without proof) in which this behaviour is different, as well as another model (or class thereof) with similar properties.

(f) [3 points] Study the influence of the parameter λ_+ and λ_- on the left wing of the smile, and provide some intuition about this result.

3 [30 points] The Ornstein-Uhlenbeck process

For a given standard Brownian motion W on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, consider the Ornstein-Uhlenbeck process, defined as

$$X_t = x\mathrm{e}^{\mu t} + \sigma \int_0^t \mathrm{e}^{\mu(t-s)} \mathrm{d}W_s,$$

for some $\mu, x \in \mathbb{R}$ and $\sigma > 0$. This process is widely used in the finance, to model, for example, the dynamics of the short rate or the evolution of the instantaneous volatility on Equity markets.

(i) We first consider the properties of the process $(X_t)_{t\geq 0}$.

(a) [3 points] Prove that the process $(X_t)_{t\geq 0}$ satisfies the stochastic differential equation

$$\mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma \mathrm{d}W_t, \qquad X_0 = x. \tag{2}$$

Show that, for any $t \geq 0$, X_t is a Gaussian random variable with

$$\mathbb{E}(X_t) = x\mathrm{e}^{\mu t} \qquad \text{and} \qquad \mathbb{V}(X_t) = \frac{\sigma^2}{2\mu}\left(\mathrm{e}^{2\mu t} - 1\right).$$

(b) [2 points] Show that the SDE (2) admits a unique strong solution.

(c) [4 points] What is the distribution of the random variable $\int_0^T X_t \mathrm{d}t$?

(ii) Introduce now the function $u : [0, T] \times \mathbb{R} \to \mathbb{R}$ as the solution to the heat equation

$$\left(\partial_t + \frac{1}{2}\partial_{xx}\right) u(t, x) = 0, \qquad \text{for all } (t, x) \in [0, T) \times \mathbb{R}, \tag{3}$$

with boundary condition $u(T, x) \equiv f(x)$ on $\mathbb{R}$, where f is a continuous function satisfying some growth conditions ensuring that all the integrals below are well defined.

(a) [4 points] Show that the process $u(t, W_t)_{t\in[0,T]}$ is a local martingale.

(b) [4 points] Assuming that $u(\cdot)$ is bounded, prove that the general solution to the PDE (3) reads

$$u(t, x) = \int_{\mathbb{R}} \frac{f(x + y\sqrt{T-t})}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) \mathrm{d}y.$$

(c) [5 points] Using this and Part (1), determine explicitly the unique bounded solution to the PDE

$$\left(\partial_t + x\partial_x + \frac{1}{2}\partial_{xx}\right) u(t, x) = 0, \qquad \text{for } (t, x) \in [0, T) \times \mathbb{R},$$

with boundary condition $u(T, x) \equiv x$ on $\mathbb{R}$.

(iii) We now wish to compute the characteristic function Φ of X_T:

$$\Phi_T(\xi) := \mathbb{E}\left(\mathrm{e}^{\mathtt{i}\xi X_T} | X_0 = x\right), \qquad \text{for all } \xi \in \mathbb{R}.$$

(a) [3 points] Fixing $\xi \in \mathbb{R}$, show that the function $v : [0,T] \times \mathbb{R} \to \mathbb{R}$ defined as $v(t,x) := \mathbb{E}\left(\mathrm{e}^{\mathtt{i}\xi X_T} | X_t = x\right)$, satisfies the partial differential equation

$$\left(\partial_t + \mu x \partial_x + \frac{1}{2}\sigma^2 \partial_{xx}\right) v(t,x) = 0, \qquad \text{for all } (t,x) \in [0,T) \times \mathbb{R}, \tag{4}$$

with boundary conditions to determine.

(b) [5 points] Using the ansatz $v(t,x) = \exp\Big(\beta(t) + \mathtt{i}\xi\alpha(t)x\Big)$, for some functions $\alpha(\cdot)$ and $\beta(\cdot)$, prove the identity

$$\Phi_T(\xi) = \exp\left(\mathtt{i}\xi x \mathrm{e}^{\mu T} - \frac{\sigma^2\xi^2}{4\mu}\left(\mathrm{e}^{2\mu T} - 1\right)\right), \qquad \text{for all } \xi \in \mathbb{R},$$

and compare it with the result obtained in (i)(a).

4 [30 points] Pricing with strict local martingales

We consider a financial market where interest rates are null, and where a given stock price satisfies the following stochastic differential equation:

$$\mathrm{d}S_t = S_t^2 \mathrm{d}W_t, \qquad S_0 = 1, \tag{5}$$

where $(W_t)_{t\geq 0}$ is a standard Brownian motion. We wish to investigate some of the mathematical properties of this process and to study the financial implications. We shall denote by $\mathcal{N}$ the Gaussian cumulative distribution function, and fix a time horizon $T > 0$.

(i) A trader is interested in the financial claim with payoff $\phi_T := S_T$.

(a) [3 points] Show that there exists a replicating trading strategy, with wealth $\phi_t = u(t, S_t)$, where

$$u(t,s) = s\left(2\mathcal{N}\left(\frac{1}{s\sqrt{T-t}}\right) - 1\right).$$

(b) [3 points] Consider the portfolio Π consisting of buying S_0 claims and selling ϕ_0 shares:

$$\Pi_t = S_0\phi_t - \phi_0 S_t, \qquad \text{for any } t \in [0,T].$$

Show that $\Pi_0 = 0$ and $\Pi_T > 0$. Prove by contradiction that this arbitrage is not admissible.

(ii) Let $\mathrm{Z} = (Z^1, Z^2, Z^3)$ denote a standard Brownian motion in $\mathbb{R}^3$, and fix $\mathrm{z} = (1,0,0) \in \mathbb{R}^3$.

(a) [4 points] Define the process $(X_t)_{t\geq 0}$ pathwise by $X_t := \|\mathrm{Z}_t - \mathrm{z}\|^{-1}$, where, for $\mathrm{x} = (x_1, x_2, x_3)$, $\|\mathrm{x}\| := (x_1^2 + x_2^2 + x_3^2)^{1/2}$ denotes the Euclidean norm in $\mathbb{R}^3$. Prove that there exists a one-dimensional Brownian motion B such that $\mathrm{d}X_t = X_t^2 \mathrm{d}B_t$, with $X_0 = 1$, and deduce that $(X_t)_{t\geq 0}$ is a positive local martingale.

(b) [3 points] Let $(x,y,z) \in \mathbb{R}^3$ be a system of Cartesian coordinates. Recall that the corresponding spherical coordinates are defined as

$$(r,\theta,\phi) = \left(\sqrt{x^2+y^2+z^2}, \operatorname{atan}\left(\frac{y}{x}\right), \operatorname{acos}\left(\frac{z}{r}\right)\right),$$

with $r \geq 0$, $\theta \in [0, 2\pi)$ and $\phi \in [0,\pi]$. Show that the inverse mapping is given by

$$(x,y,z) = \left(r\cos(\theta)\sin(\phi), r\sin(\theta)\sin(\phi), r\cos(\phi)\right).$$

(c) [7 points] Deduce from this and the definition of X that $\mathbb{E}(X_t) = 2\mathcal{N}\left(1/\sqrt{t}\right) - 1$; what does that imply for the process X? How does the Put-Call parity look like at strike zero?

(iii) We finally propose an alternative proof to the result in the previous item.

(a) [3 points] Let Y be a positive martingale starting from one, and define the measure $\mathbb{Q}$ by $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} := Y_T$. Show that the process $(X_t)_{t\in[0,T]} := (Y_t^{-1})_{t\in[0,T]}$ is a positive martingale under $\mathbb{Q}$.

(b) [3 points] If Y satisfies $\mathrm{d}Y_t = \sigma_t Y_t \mathrm{d}W_t$, for some Brownian motion W and some adapted process σ, prove that X satisfies $\mathrm{d}X_t = \sigma_t X_t \mathrm{d}W_t^{\mathbb{Q}}$ for some $\mathbb{Q}$-Brownian motion $W^{\mathbb{Q}}$.

(c) [4 points] Letting $\sigma_\cdot = Y_\cdot$, prove that the solution to (5) satisfies

$$\mathbb{P}(S_t > 0) = 1 \qquad \text{and} \qquad \mathbb{Q}(S_t > 0) = \mathcal{N}(1/\sqrt{t}),$$

for all $t \in [0,T]$, and deduce from this that S is a strict local martingale.