M5MF6, EXERCICE SET: STOCHASTIC ANALYSIS AND BROWNIAN MOTION

We shall here consider a given filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, supporting a standard Brownian motion $(W_t)_{t\geq 0}$, with natural filtration $(\mathcal{F}_t)_{t\geq 0}$.

Exercise 1

Prove Proposition 1.1.3, Theorem 1.2.9, and the examples/exercises in the lecture notes in Sections 1.1-1.2.

Solution to Exercise 1

See the lecture notes.

Exercise 2 An optional time which is not a stopping time

Consider the filtration $(\mathcal{G}_t)_{t\geq}$ and the random time τ defined by

$$\mathcal{G}_t := \begin{cases} \{\emptyset, \Omega\}, & \text{if } t \leq 1, \\ 2^{\Omega}, & \text{if } t > 1. \end{cases} \qquad \text{and} \qquad \tau := \begin{cases} 1, & \text{if } \omega \in B, \\ 2, & \text{if } \omega \notin B, \end{cases}$$

where B is some non-trivial subset of Ω. Show that τ is a $\mathcal{G}$-optional time, but not a $\mathcal{G}$-stopping time.

Exercise 3 Gaussian moments

Let $(B_t)_{t\geq 0}$ denote a one-dimensional standard Brownian motion on the real line, and, for any $n \in \mathbb{N}$, define $\beta_n(t) := \mathbb{E}(B_t^n)$. Show, using Ito's formula, that the identity

$$\beta_n(t) = \frac{1}{2}n(n-1)\int_0^t \beta_{n-2}(s)\mathrm{d}s$$

holds for all $t \geq 0$ and $n \geq 2$, and deduce that, for all $t \geq 0$,

$$\beta_n(t) = \begin{cases} 0, & \text{if } n \text{ is odd}, \\ \dfrac{(2m)!t^m}{2^m m!}, & \text{if } n \text{ is even with } n = 2m. \end{cases}$$

Exercise 4 Gaussian integral

Define the process $(X_t)_{t\geq 0}$ pathwise by $X_t := \int_0^t \varphi_s \mathrm{d}W_s$, where φ is a deterministic function (path). Prove that X is a Gaussian process with mean zero and covariance structure $\mathbb{E}(X_s X_t) = \int_0^{s\wedge t} \varphi_s^2 \mathrm{d}s$.

Solution to Exercise 4

We first prove that, for any $t \geq 0$, X_t is a Gaussian random variable with mean zero and variance $\int_0^t \varphi_s^2 \mathrm{d}s$. Itô's lemma yields

$$\mathbb{E}\left(\mathrm{e}^{uX_t}\right) = 1 + \frac{1}{2}u^2 \int_0^t \varphi_s^2 \mathbb{E}\left(\mathrm{e}^{uX_s}\right)\mathrm{d}s, \tag{0.1}$$

Date: January 19, 2017.

for any real number u. For any such u, define $\psi_t := \mathbb{E}\left(\mathrm{e}^{uX_t}\right)$, so that differentiating both sides of (0.1) implies

$$\partial_t \psi_t = \frac{1}{2} u^2 \varphi_t^2 \psi_t,$$

which is easily solved as $\psi_t = \exp\left(\frac{1}{2}u^2 \int_0^t \varphi_s^2 \mathrm{d}s\right)$. This proves the claim. The proof of the covariance structure follows the same steps.

Exercise 5 Complex Brownian motion

Given a two-dimensional Brownian motion $(B_t^{(1)}, B_t^{(2)})_{t\geq 0}$, define the complex Brownian motion

$$\mathrm{B}_t := B_t^{(1)} + \mathtt{i} B_t^{(2)},$$

where $\mathtt{i}^2 = -1$. Let $f : \mathbb{C} \to \mathbb{C}$ be a function of the form $f(z) = f_{\mathbb{R}}(z) + \mathtt{i} f_{\mathbb{C}}(z)$, for any $z \in \mathbb{C}$, with $f_{\mathbb{R}}, f_{\mathbb{C}} : \mathbb{C} \to \mathbb{R}$. If f is analytic, i.e. satisfies the Cauchy-Riemann equations

$$\frac{\partial f_{\mathbb{R}}}{\partial x} = \frac{\partial f_{\mathbb{C}}}{\partial y} \qquad \text{and} \qquad \frac{\partial f_{\mathbb{R}}}{\partial y} = -\frac{\partial f_{\mathbb{C}}}{\partial x},$$

where $z = x + \mathtt{i}y$, show that the identity

$$\mathrm{d}f(\mathrm{B}_t) = f'(\mathrm{B}_t)\mathrm{d}\mathrm{B}_t$$

holds almost surely for all $t \geq 0$, where f' denotes the complex derivative of f. As an example, solve explicitly the complex stochastic differential equation

$$\mathrm{d}Z_t = \alpha Z_t \mathrm{d}\mathrm{B}_t, \qquad Z_0 = z \in \mathbb{C}.$$

Solution to Exercise 5

Let $Z. := f(\mathrm{B}.)$. By linearity of the derivative operator, we can write

$$\begin{aligned}
\mathrm{d}Z_t &= \mathrm{d}f_{\mathbb{R}}(\mathrm{B}_t) + \mathtt{i}\mathrm{d}f_{\mathbb{C}}(\mathrm{B}_t) = \mathrm{d}f_{\mathbb{R}}(B_t^{(1)}, B_t^{(2)}) + \mathtt{i}\mathrm{d}f_{\mathbb{C}}(B_t^{(1)}, B_t^{(2)}) \\
&= \nabla f_{\mathbb{R}}(\mathrm{B}_t)\mathrm{d}\mathrm{B}_t + \frac{1}{2}\Delta f_{\mathbb{R}}(\mathrm{B}_t)\mathrm{d}t + \mathtt{i}\nabla f_{\mathbb{C}}(\mathrm{B}_t)\mathrm{d}\mathrm{B}_t + \frac{\mathtt{i}}{2}\Delta f_{\mathbb{C}}(\mathrm{B}_t)\mathrm{d}t \\
&= (\nabla f_{\mathbb{R}} + \mathtt{i}\nabla f_{\mathbb{C}})(\mathrm{B}_t)\mathrm{d}\mathrm{B}_t \\
&= \left(\frac{\partial f_{\mathbb{R}}}{\partial x}\mathrm{d}B_t^{(1)} + \frac{\partial f_{\mathbb{R}}}{\partial y}\mathrm{d}B_t^{(2)}\right) + \mathtt{i}\left(\frac{\partial f_{\mathbb{C}}}{\partial x}\mathrm{d}B_t^{(1)} + \frac{\partial f_{\mathbb{C}}}{\partial y}\mathrm{d}B_t^{(2)}\right) \\
&= \left(\frac{\partial f_{\mathbb{R}}}{\partial x}\mathrm{d}B_t^{(1)} - \frac{\partial f_{\mathbb{C}}}{\partial x}\mathrm{d}B_t^{(2)}\right) + \mathtt{i}\left(\frac{\partial f_{\mathbb{C}}}{\partial x}\mathrm{d}B_t^{(1)} + \frac{\partial f_{\mathbb{R}}}{\partial y}\mathrm{d}B_t^{(2)}\right) \\
&= \left(\frac{\partial f_{\mathbb{R}}}{\partial x}\mathrm{d}B_t^{(1)} + \frac{\partial f_{\mathbb{R}}}{\partial y}\mathrm{d}B_t^{(2)}\right) + \mathtt{i}\left(\frac{\partial f_{\mathbb{C}}}{\partial x}\mathrm{d}B_t^{(1)} + \mathtt{i}\frac{\partial f_{\mathbb{C}}}{\partial x}\mathrm{d}B_t^{(2)}\right) = f'(\mathrm{B}_t)\mathrm{d}\mathrm{B}_t.
\end{aligned}$$

Using the complex Ito formula above, we immediately obtain $\mathrm{d}\mathrm{e}^{\alpha\mathrm{B}_t} = \alpha\mathrm{e}^{\alpha\mathrm{B}_t}\mathrm{d}\mathrm{B}_t$, so that the process defined by $Z_t = Z_0 + \mathrm{e}^{\alpha\mathrm{B}_t}$ for all $t \geq 0$, solves the complex SDE.

Exercise 6 Pinned Brownian motion

Find a Borel function ϕ such that $\mathbb{E}(W_s|W_t) = \phi(W_t)$, for any $0 \leq s \leq t$.

Solution to Exercise 6

Fix $0 \leq s < t$. We are looking here for a Borel function ϕ such that $\mathbb{E}(W_s|W_t) = \phi(W_t)$; we can write this equivalently, for any Borel subset $\mathcal{B} \subset \mathbb{R}$, as

$$\int_{\mathcal{B}} W_s \mathrm{d}\mathbb{P}_t(\mathrm{d}x) = \int_{\mathcal{B}} \varphi(x)\mathbb{P}_t(\mathrm{d}x) = \int_{\mathcal{B}} \varphi(x) p_t(0,x)\mathrm{d}x,$$

where $\mathbb{P}_t$ is the law of the Brownian motion at time t, and p_t the Gaussian density at time t; furthermore,

$$\begin{aligned}\int_{\mathcal{B}} W_s \mathbb{P}_t(\mathrm{d}x) &= \int_{\mathcal{B}} \left(\int_{\mathbb{R}} x p_s(0,x) p_{t-s}(x,y)\mathrm{d}x\right)\mathrm{d}y \\ &= \int_{\mathcal{B}} \left(p(t;0,y)\int_{\mathbb{R}} x p_{s(t-s)/t}\left(\frac{sy}{t},x\right)\mathrm{d}x\right)\mathrm{d}y \\ &= \int_{\mathcal{B}} \frac{sy}{t} p_t(0,y)\mathrm{d}y.\end{aligned}$$

Taking the function $\varphi(y) \equiv sy/t$ concludes the proof.

Exercise 7 Supremum of Brownian motion and Quadratic variation

Let M be a continuous local martingale starting at the origin. Then, for all $x, u > 0$,

$$\mathbb{P}\left(\sup_{t\geq 0} M_t > x, \langle M\rangle_\infty \leq u\right) \leq \exp\left(-\frac{x^2}{2u}\right).$$

Hint: Fix some $x > 0$, and let $\tau := \inf\{t \geq 0 : M_t \geq x\}$ be the first hitting time of the level x. For any $\alpha \in \mathbb{R}$, introduce the process $(Z_t)_{t\geq 0}$ defined pathwise by

$$Z_t := \exp\left(\alpha M_t^\tau - \frac{1}{2}\alpha^2 \langle M\rangle_t^\tau\right).$$

Use then the optional sampling theorem: a martingale stopped at a stopping time remains a martingale.

Solution to Exercise 7

Fix some $x > 0$, and let $\tau := \inf\{t \geq 0 : M_t \geq x\}$ be the first hitting time of the level x. For any $\alpha \in \mathbb{R}$, introduce the process $(Z_t)_{t\geq 0}$ defined pathwise by

$$Z_t := \exp\left(\alpha M_t^\tau - \frac{1}{2}\alpha^2 \langle M\rangle_t^\tau\right).$$

Clearly, Z is a continuous local martingale (by Itô's formula), and satisfies the inequality $|Z_t| \leq \exp(\alpha x)$ almost surely for all $t \geq 0$. When $\alpha = 1$, Z is usually called the stochastic exponential of M, and is denoted by $\mathcal{E}(M)$. It is therefore square integrable, and the optional sampling theorem implies $\mathbb{E}(Z_\infty) = \mathbb{E}(Z_0) = 1$. Markov's inequality further yields, for any $u > 0$,

$$\mathbb{P}\left(\sup_{t\geq 0} M_t > x, \langle M\rangle_\infty \leq u\right) \leq \mathbb{P}\left(Z_\infty \geq \exp\left(\alpha x - \frac{1}{2}\alpha^2 u\right)\right) \leq \exp\left(-\alpha x + \frac{1}{2}\alpha^2 u\right).$$

Since α is taken randomly, one can optimise over it, and the maximum on the right-hand side is clearly attained at $\alpha = x/u$, from which the result follows.

Exercise 8 Martingale Representation Theorem

Write down the explicit form of the martingale representation theorem for the process $(M_t)_{t\geq 0}$ defined as

(i) $M_t = \int_0^t W_s \mathrm{d}s$;
(ii) $M_t = W_t^2$;
(iii) $M_t = \int_0^t W_s^2 \mathrm{d}s$;
(iv) $M_t = \sin(W_t)$;
(v) $M_t = \exp(W_t)$.

Solution to Exercise 8

(i) Clearly, $M_t = \int_0^t \mathrm{d}W_s$;
(ii) Integration by parts immediately gives $M_t = \int_0^t (t-s)\mathrm{d}W_s$;
(iii) Itô's formula yields $M_t = t + 2\int_0^t W_s \mathrm{d}W_s = \mathbb{E}(M_t) + 2\int_0^t W_s \mathrm{d}W_s$;
(iv) Itô's formula applied to the function $f : (W,t) \mapsto tW^2$ yields

$$tW_t^2 = \int_0^t \left(W_s^2 \mathrm{d}s + 2sW_s \mathrm{d}W_s + s\mathrm{d}s\right)$$

so that

$$\begin{aligned} M_t = \int_0^t W_s^2 \mathrm{d}s &= tW_t^2 - 2\int_0^t sW_s \mathrm{d}W_s - \int_0^t s\mathrm{d}\langle W,W\rangle_s \\ &= t\left(t + 2\int_0^t W_s \mathrm{d}W_s\right) - 2\int_0^t sW_s \mathrm{d}W_s - \frac{1}{2}t^2 \\ &= \frac{1}{2}t^2 + 2\int_0^t (t-s)W_s \mathrm{d}W_s = \mathbb{E}(M_t) + 2\int_0^t (t-s)W_s \mathrm{d}W_s \end{aligned}$$

(v) Itô's formula yields

$$\begin{aligned} \sin(W_t)\exp\left(\frac{t}{2}\right) &= \int_0^t \cos(W_u)\exp\left(\frac{u}{2}\right)\mathrm{d}W_u + \frac{1}{2}\int_0^t \sin(W_u)\exp\left(\frac{u}{2}\right)\mathrm{d}u - \frac{1}{2}\int_0^t \sin(W_u)\exp\left(\frac{u}{2}\right)\mathrm{d}\langle W,W\rangle_u \\ &= \int_0^t \cos(W_u)\exp\left(\frac{u}{2}\right)\mathrm{d}W_u. \end{aligned}$$

Since $\mathbb{E}(M_t)$ is clearly null, the representation follows.
(vi) Hint: apply Itô's formula to the function $(t, W_t) \mapsto \mathrm{e}^{W_t} f(t)$, for some smooth function $f : [0,\infty) \to \mathbb{R}$.

Exercise 9 Martingale Representation Theorem 2

Define the process M by $M_t := \mathbb{E}(W_T^3|\mathcal{F}_t)$, for any $t \in [0,T]$. Prove that

$$M_t = 3\int_0^t \left(T - s + W_s^2\right)\mathrm{d}W_s.$$

Solution to Exercise 9

$$M_t = \mathbb{E}_t(W_T^3) = \mathbb{E}_t[(W_T - W_t + W_t)^3] = \mathbb{E}_t[(W_T - W_t)^3] + W_t^3 + 3W_t\mathbb{E}_t[(W_T - W_t)^2] + 3W_t^2\mathbb{E}_t[(W_T - W_t)] = W_t^3 + 3(T-t)W_t$$

Applying Itô's lemma then yields

$$\mathrm{d}M_t = 3\left(W_t^2 + T - t\right)\mathrm{d}W_t,$$

which concludes the proof.

Exercise 10

Prove that the process $Y := \left(\int_0^t W_u \mathrm{d}u\right)_{t\geq 0}$ is Gaussian, and compute its expectation and variance.

Solution to Exercise 10

The integration is defined pathwise as a Riemann integral since the integrand is continuous. Since W is Gaussian, for any $t \geq 0$, the random variable Y_t is clearly Gaussian as limit of Riemann sums, and so is the process Y. Clearly $\mathbb{E}(Y_t) = 0$, and, for any $0 \leq s \leq t$,

$$\mathbb{E}\left(Y_s Y_t\right) = \mathbb{E}\left(\int_0^s W_u \mathrm{d}u \int_0^t W_v \mathrm{d}v\right) = \int_0^s \int_0^t (u \wedge v)\mathrm{d}u\mathrm{d}v = \frac{s^2}{6}(3t - s).$$

Exercise 11

Consider the process Y defined, for all $t \geq 0$, by

$$Y_t := W_t - \int_0^t \frac{W_u}{u}\mathrm{d}u.$$

Prove that Y is Gaussian, and compute its expectation and variance. Show that it is not an $(\mathcal{F}_t)$-martingale.

Solution to Exercise 11

The process Y is the sum of two Gaussian, but these are not independent. However, the integral is a Gaussian process by definition of (sums in L^2 of) Riemannn integrals. Clearly $\mathrm{e}(Y_t) = 0$ and, for any $s, t \geq 0$,

$$\begin{aligned}
\mathbb{E}(Y_s Y_t) &= \mathbb{E}\left\{\left(W_t - \int_0^t \frac{W_u}{u}\mathrm{d}u\right)\left(W_s - \int_0^s \frac{W_v}{v}\mathrm{d}v\right)\right\} \\
&= \mathbb{E}(W_t W_s) - \mathbb{E}\left(W_t \int_0^s \frac{W_v}{v}\mathrm{d}v\right) - \mathbb{E}\left(W_s \int_0^t \frac{W_u}{u}\mathrm{d}u\right) + \mathbb{E}\left\{\left(\int_0^t \frac{W_u}{u}\mathrm{d}u\right)\left(\int_0^s \frac{W_v}{v}\mathrm{d}v\right)\right\} \\
&= s \wedge t - \int_0^s \frac{\mathbb{E}\left(W_t W_v\right)}{v}\mathrm{d}v - \int_0^t \frac{\mathbb{E}\left(W_s W_u\right)}{u}\mathrm{d}u + \int_0^t \int_0^s \frac{\mathbb{E}(W_u W_v)}{uv}\mathrm{d}u\mathrm{d}v \\
&= s \wedge t - \int_0^s \frac{t \wedge v}{v}\mathrm{d}v - \int_0^t \frac{s \wedge u}{u}\mathrm{d}u + \int_0^t \int_0^s \frac{u \wedge v}{uv}\mathrm{d}u\mathrm{d}v = s \wedge t,
\end{aligned}$$

which proves that the Gaussian process Z is a Brownian motion. However, for any $s < t$,

$$\mathbb{E}\left(Z_t - Z_s | \mathcal{F}_s^W\right) = \mathbb{E}\left(W_t - \int_0^t \frac{W_u}{u}\mathrm{d}u - W_s - \int_0^s \frac{W_u}{u}\mathrm{d}u \middle| \mathcal{F}_s^W\right) = \int_s^t \frac{W_u}{u}\mathrm{d}u,$$

which is clearly non zero almost surely.

Exercise 12 Clark-Ocone Formula

Let f be a bounded $\mathcal{C}^1$ function on $\mathbb{R}$. Prove that there exists a function $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ such that

$$\mathbb{E}\left(f(W_1)|\mathcal{F}_t\right) = g(t, W_t), \qquad \text{for any } t \in [0.1],$$

and write down an Itô formula for g. Prove finally that the following equality holds for all $t \in [0,1]$:

$$g(t, W_t) = \mathbb{E}(f(W_1)) + \int_0^t \mathbb{E}\left(f'(W_1|\mathcal{F}_s\right)\mathrm{d}W_s.$$

Solution to Exercise 12

It is easy to see that

$$\mathbb{E}\left(f(W_1)|\mathcal{F}_t\right) = \mathbb{E}\left(f(W_1 - W_t + W_t)|\mathcal{F}_t\right) = \mathbb{E}\left[f\left(\widehat{W}_{1-t} + W_t\right)\middle|\mathcal{F}_t\right] = g(t, W_t),$$

where the function g is defined as $g(t,x) \equiv \mathbb{E}\left(f(x+\widehat{W}_{1-t})|\mathcal{F}_t\right)$. Note that the process $(g(t,W_t))_{t\in[0,1]}$ is clearly a martingale, and hence

$$g(t, W_t) = \mathbb{E}(f(W_1)) + \int_0^t \partial_x g(s, W_s)\mathrm{d}W_s = \mathbb{E}(f(W_1)) + \int_0^t \mathbb{E}\left[f'(W_s + \widehat{W}_{1-s})|\mathcal{F}_s\right]\mathrm{d}W_s,$$

and the result follows.

Department of Mathematics, Imperial College London
E-mail address: `a.jacquier@imperial.ac.uk`