M5MF6, EXERCICE SET - REVISIONS

Exercise 1

Let $b, \sigma : \mathbb{R} \to \mathbb{R}$ be smooth and globally Lipschitz. Prove that the solution to the SDE

$$\mathrm{d}X_t^{0,x} = b(X_t^{0,x})\mathrm{d}t + \sigma(X_t^{0,x})\mathrm{d}W_t, \qquad X_0^{0,x} = x \in \mathbb{R},$$

is time-homogeneous, i.e. that $(X_{t+h}^{t,x})$ and $(X_h^{0,x})_{h\geq 0}$ have the same law for any $s, t \geq 0$.

Solution to Exercise 1

We can write

$$\begin{aligned} X_{t+h}^{t,x} &= x + \int_t^{t+h} b(X_u^{t,x})\mathrm{d}u + \int_t^{t+h} \sigma(X_u^{t,x})\mathrm{d}W_u \\ &= x + \int_0^h b(X_{t+v}^{t,x})\mathrm{d}v + \int_0^h \sigma(X_{t+v}^{t,x})\mathrm{d}B_v, \end{aligned}$$

with the change of variables $u = t + v$, and where $B_v := W_{t+v} - W_t$. Now, since

$$X_t^{0,x} = x + \int_0^h b(X_v^{0,x})\mathrm{d}v + \int_0^h \sigma(X_v^{0,x})\mathrm{d}W_v,$$

and using the facts that both W and B have the same distributions and the SDE has a weak solution, the exercise follows.

Exercise 2 (Strong) Markovianity

Show that the following processes are Markovian, but not strong Markovian:

(1) Let X be a continuous Markov process. If $X_0 = 0$, then $X_t = 0$ almost surely for all $t \geq 0$. If $X_0 \neq 0$, then X is a standard Brownian motion starting from X_0.

(2) Let W be a standard Brownian motion and define $X_t := f(W_t + \pi)$, with

$$f(x) := \begin{cases} (x, 0), & \text{if } x \leq 0, \\ (\sin(x), 1 - \cos(x)), & \text{if } x \in (0, 2\pi), \\ (x - 2\pi, 0), & \text{if } x \geq 2\pi. \end{cases}$$

Hint: To show that X is Markov, use the fact that φ is invertible except at the origin. To show that X is not strong Markov, consider the hitting time of the origin.

Remark 0.1. The curve given by f intersects itself so, if you stop at an intersection point, you do not know which part of the curve the process is currently moving on restart.

Remark 0.2. In the discrete-time setting, Markov and strong Markov are equivalent.

Exercise 3

Consider the Black-Scholes stochastic differential equation $\mathrm{d}S_t = S_t(\mu\mathrm{d}t + \sigma\mathrm{d}W_t)$, for $\mu \in \mathbb{R}$ and $\sigma > 0$, where W is a standard Brownian motion.

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(i) For any $t \geq 0$, compute the distribution of the random variable S_t.

(ii) Using probabilistic arguments only (and (i)), compute the price $\mathbb{E}\left(S_T - K\right)_+$ at inception of a standard European Call option with maturity $T > 0$ and strike $K \geq 0$.

Exercise 4 Ornstein-Uhlenbeck process conditioned to stay positive

Let X be an Ornstein-Uhlenbeck process, solution to $\mathrm{d}X_t = -X_t\mathrm{d}t + \mathrm{d}W_t$, starting from $X_0 = x > 0$. Let the random variable Y defined via its distribution

$$\mathbb{P}(Y \in \mathrm{d}y) := \lim_{t\uparrow\infty} \mathbb{P}\left(X_t \in \mathrm{d}y \middle| \sup_{0\leq s\leq t} X_s \geq 0\right), \qquad \text{for any } y \in \mathbb{R}.$$

Show that there exist strictly positive constants α and β such that

$$\mathbb{P}(Y \geq x) \leq \alpha \mathrm{e}^{-\beta x^2} \qquad \text{for all } x \in \mathbb{R}.$$

Hint: Defining the function $\varphi(t) \equiv \mathrm{e}^{2t}-1$, use the fact that the process X can be written as $X_t = x\mathrm{e}^{-t}+\mathrm{e}^{-t}W_{\varphi(t)}$.

Solution to Exercise 4

Using time-change techniques, we can write

$$X_t = x\mathrm{e}^{-t} + \mathrm{e}^{-t}W_{\varphi(t)}, \qquad \text{for all } t \geq 0,$$

where $\varphi(t) \equiv \mathrm{e}^{2t} - 1$. For any $t \geq 0$, define now the events A_t and B_t as

$$A_t := \left\{W_{\varphi(t)} > y\mathrm{e}^t - x\mathrm{e}^{-t}\right\} \qquad \text{and} \qquad B_t := \left\{W_{\varphi(s)} > -x\mathrm{e}^{-s}, s \leq t\right\}.$$

We wish to compute $\mathbb{P}(A_t|B_t) = \mathbb{P}(A_t \cap B_t)/\mathbb{P}(B_t)$. The following inequality clearly holds from the reflection principle for Brownian motion:

$$\mathbb{P}(B_t) \leq \mathbb{P}\left(W_s > -x\mathrm{e}^{-1}, s \in [0,1]; W_1 > 1; W_s - W_1 \geq -1, s \in (1, \varphi(t))\right) \geq -C(x)\mathrm{e}^{-t}.$$

Likewise, for t large enough and $y > 2$, we can write

$$\mathbb{P}(A_t \cap B_t) \leq \mathbb{P}\left(W_s \geq -x, s \leq \mathrm{e}^{2t} - 1, W_{\mathrm{e}^{2t}-1} \geq (y-1)\mathrm{e}^t\right) \leq C(x)\mathrm{e}^{-t}y\mathrm{e}^{-y^2/2}.$$

Exercise 5 Brownian bridge

Let $(X_t)_{t\geq 0}$ be the unique strong solution to the following SDE, starting at $X_0 = 0$:

$$\mathrm{d}X_t = \frac{X_t}{t-1}\mathrm{d}t + \mathrm{d}B_t, \quad \text{for } 0 \leq t < 1.$$

(i) Show that, for any $t \in [0, 1)$, the equality $X_t = (1-t)\displaystyle\int_0^t \frac{\mathrm{d}W_s}{1-s}$ holds almost surely;

(ii) Show that X is a Gaussian process and compute its expectation and covariance function;

(iii) Show that $\lim_{t\uparrow 1} X_t = 0$ almost surely.

Solution to Exercise 5

(i) Let $Y_t := \int_0^t \frac{\mathrm{d}W_s}{1-s}$. Apply Itô's lemma to the function $(t, Y_t) \mapsto (1-t)Y_t$.

(ii) X is Gaussian since it is a Wiener integral and clearly $\mathbb{E}(X_t) = 0$. The computation of the covariance is straightforward and $\mathbb{E}(X_sX_t) = s(1-t)$ whenever $s < t$.

(iii) In particular, $\mathbb{E}(X_t^2) = t(1-t)$ which converges to zero as t tends to 1, and the result follows.

Exercise 6

Consider the stochastic differential equation, for any integer n:

$$\mathrm{d}X_t = (\alpha X_t^n + \beta X_t)\,\mathrm{d}t + \gamma X_t \mathrm{d}W_t, \qquad X_0 = x \in \mathbb{R},$$

Using the substitution $Y_t := X_t^{1-n}$, show that the solution reads

$$X_t = Z_t \left(X_0^{1-n} + \alpha(1-n) \int_0^t Z_s^{n-1} \mathrm{d}s \right)^{1/(1-n)},$$

where $Z_t := \exp\left\{ \left(\beta - \frac{1}{2}\gamma^2\right) t + \gamma W_t \right\}$.

Remark 0.3. The case $n = 2$ with $\beta = -1$ is called the stochastic Verhulst equation, and is used in population dynamics. For $n = 2$ with $\alpha = -1$ is called the stochastic Ginzburg-Landau equation, a fundamental equation in physics.

Solution to Exercise 6

The exercise follows from Itô's formula.

Exercise 7 Linear SDE

Let W be a d-dimensional Brownian motion, ξ an independent random variable with values in $\mathbb{R}^n$, and the following linear stochastic differential equation:

$$X_t = \xi + \int_0^t \left[b_0(s) + b_1(s) X_s \right] \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}W_s, \quad \text{for all } t \geq 0,$$

where b_0 and b_1 are measurable functions from $\mathbb{R}_+$ to $\mathbb{R}^n$, and $(\sigma_t)_{t\geq 0}$ a process adapted to the Brownian filtration and taking values in $\mathcal{M}_{d,n}(\mathbb{R})$. All the coefficients are assumed to satisfy the usual conditions ensuring a unique strong solution.

Let $\Phi : \mathbb{R}_+ \to \mathcal{M}_{n,n}(\mathbb{R})$ be the unique solution to the ordinary differential equation

$$\Phi_t = I_n + \int_0^t b_1(s) \Phi_s \mathrm{d}s, \quad \text{for all } t \geq 0,$$

where I_n denotes the identity matrix. Prove that X admits the closed-form representation:

$$X_t = \Phi_t \left(\xi + \int_0^t \Phi_s^{-1} b_0(s) \mathrm{d}s + \int_0^t \Phi_s^{-1} \sigma_s \mathrm{d}W_s \right)^{-1}, \quad \text{for all } t \geq 0.$$

Solution to Exercise 7

The exercise follows by a direct application of Itô's formula.

Exercise 8 Ornstein-Uhlenbeck process

Consider the mean-reverting Orstein-Uhlenbeck model:

$$\mathrm{d}X_t = \kappa(\theta - X_t)\mathrm{d}t + \xi \mathrm{d}W_t,$$

where W is a standard one-dimensional Brownian motion, and X_0 a Gaussian random variable independent of W, with mean μ and variance σ^2. Show that, for any $t \geq 0$,

$$X_t = X_0 \mathrm{e}^{-\kappa t} + \theta\left(1 - \mathrm{e}^{-\kappa t}\right) + \xi \int_0^t \mathrm{e}^{-\kappa(t-s)} \mathrm{d}W_s, \qquad \text{for all } t \geq 0.$$

Compute the probability $\mathbb{P}(X_t < 0)$, and discuss potential issues with using this model as the instantaneous volatility in a stochastic volatility model (as originally proposed by E. Stein and J. Stein in 1991).

Solution to Exercise 8

The first part of the exercise is a straightforward application of Itô's lemma. You can, for a start, consider the transformed process Y defined by $Y_t = \mathrm{e}^{\kappa t} X_t$. For any $t \geq 0$, X_t is a Gaussian random variable, and therefore $\mathbb{P}(X_t < 0)$ is strictly positive. One could argue that volatility should be a strictly positive quantity. However, note that in the SDE $\mathrm{d}S_t = X_t S_t \mathrm{d}W_t$, the product $X_t W_t$ can be either negative or positive, since $\mathrm{d}W_t$ takes values on the whole real line, so this may not be an issue after all, apart from intuitive conventions.

Exercise 9 Feynman-Kac

Consider the arithmetic Brownian motion with drift:

$$\mathrm{d}X_t = \alpha \mathrm{d}t + \sigma \mathrm{d}W_t.$$

The goal of the exercise is to use Feynman-Kac formula in order to compute the characteristic function $u(t, x) := \mathbb{E}(u(X_T)|X_t = x)$, with $u(x) \equiv \mathrm{e}^{\mathtt{i}\xi x}$. Prove that the function u satisfies the partial differential equation $\mathcal{L}u(t, x) = 0$, for all $t \in [0, T)$, $x \in \mathbb{R}$, with some boundary conditions, and where the differential operator $\mathcal{L}$ needs to be written out explicitly.

Assuming that $u(t, x)$ can be written as the product $\phi(t)\psi(x)$, for some functions ϕ and ψ, show that

$$u(t, x) = \exp\left\{\mathtt{i}\left(x + \alpha(T - t)\right)\xi - \frac{\sigma^2\xi^2}{2}(T - t)\right\}.$$

Solution to Exercise 9

Feynman-Kac's formula yields the PDE $\mathcal{L}u = 0$, with boundary condition $u(T, x) \equiv \mathrm{e}^{\mathtt{i}\xi x}$, with

$$\mathcal{L} = \mu\partial_x + \frac{\sigma^2}{2}\partial_{xx} + \partial_t.$$

In order to solve the PDE, we use the separation of variables, i.e. assuming that $u(t, x)$ can be written as the product $\phi(t)\psi(x)$, we can show by direct computations that ϕ and ψ satisfy the identity

$$\frac{\alpha\psi'(x)}{\psi(x)} + \frac{\sigma^2\psi''(x)}{2\psi(x)} = -\frac{\phi'(t)}{\phi(t)}.$$

Since both sides are equal, they must be equal to a constant, say λ. Both ODEs can then be solved separately, and the boundary conditions give the value of λ.

Exercise 10 Itô and PDEs

Consider the Black-Scholes SDE

$$\mathrm{d}S_t/S_t = r\mathrm{d}t + \sigma\mathrm{d}W_t, \qquad S_0 > 0.$$

Let h be a continuous function on $\mathbb{R}_+$, and let u be the solution to the PDE

$$\left(\partial_t + rs\partial_s + \frac{\sigma^2 s^2}{2}\partial_s s\right) u(t, s) = 0,$$

with boundary condition $u(T, s) \equiv h(s)$, for some $T > 0$.

(1) Using Itô's formula, show that $u(t, S_t)$ is a local martingale;
(2) Is it a true martingale?

(3) Deduce the financial meaning of the quantity $u(0, s)$.

Solution to Exercise 10

Itô's formula yields

$$\mathrm{d}u(t, S_t) = (\cdots)\,\mathrm{d}t + \sigma \mathrm{d}W_t,$$

so that $(u(t, S_t))_{t\geq 0}$ is clearly a local martingale. Since σ is constant, the true martingality is immediate. Therefore

$$u(0, s) = \mathbb{E}\left[u(T, S_T)|S_0 = s\right] = \mathbb{E}\left[h(S_T)|S_0 = s\right],$$

so that $u(0, s)$ represents the price at inception of a European option on S, with maturity T and payoff $h(\cdot)$.

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