M5MF6, EXERCICE SET: CHANGES OF MEASURES AND OPTION PRICING

We shall here consider a given filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, supporting a standard Brownian motion $(W_t)_{t\geq 0}$, with natural filtration $(\mathcal{F}_t)_{t\geq 0}$. Whenever needed, $T > 0$ will denote a fixed time horizon.

Exercise 1

Let $(\varphi_t)t \in [0, T]$ be an adapted bounded process, and define the process $(Z_t)_{t\in[0,T]}$ as the unique solution to $\mathrm{d}Z_t = -\varphi_t Z_t \mathrm{d}W_t$, starting from $Z_0 = 1$. For any $t \geq 0$, define (the restriction to $\mathcal{F}_t$) $\mathbb{Q}$ as

$$\mathrm{d}\mathbb{Q}|_{\mathcal{F}_t} := Z_t \mathrm{d}\mathbb{P}|_{\mathcal{F}_t}.$$

Prove that

$$\mathbb{E}_{\mathbb{P}} \left(Z_T \log(Z_T) \right) = \mathbb{E}^{\mathbb{Q}} \left(\frac{1}{2} \int_0^T \varphi_s^2 \mathrm{d}s \right).$$

Solution to Exercise 1

It is clear that Z is a strictly positive $\mathbb{P}$-martingale, so that $\mathbb{Q}$ defines a genuine probability measure, and therefore

$$\mathbb{E}_{\mathbb{P}} \left(Z_T \log(Z_T) \right) = \mathbb{E}_{\mathbb{Q}} \left(\log(Z_T) \right).$$

Now, applying Itô's formula, we can write

$$Z_T = \exp \left(-\frac{1}{2} \int_0^T \varphi_s^2 \mathrm{d}s - \int_0^T \varphi_s \mathrm{d}W_s \right),$$

From Girsanov's theorem, the process $\widetilde{W}$ defined pathwise as $\widetilde{W}_t := W_t + \int_0^t \varphi_s \mathrm{d}s$ is a standard Brownian motion under $\mathbb{Q}$, and

$$-\int_0^t \varphi_s \mathrm{d}W_s - \frac{1}{2} \int_0^t \varphi_s^2 \mathrm{d}s = -\int_0^t \varphi_s \mathrm{d}\widetilde{W}_s + \frac{1}{2} \int_0^t \varphi_s^2 \mathrm{d}s,$$

from which the result follows.

Exercise 2

Let $(X_t)_{t\geq 0}$ be the unique solution to the following stochastic differential equation, under $\mathbb{P}$:

$$\mathrm{d}X_t = X_t \left(\mu_t \mathrm{d}t + \sigma_t \mathrm{d}W_t \right), \qquad X_0 = 1,$$

where μ and σ are bounded and adapted processes, and $\sigma > 0$ almost surely.

(i) Show that $X_t \exp \left(-\int_0^t \mu_s \mathrm{d}s \right)$ is a local martingale.
(ii) Find a probability $\mathbb{Q}$, equivalent to $\mathbb{P}$, under which X is a local martingale.
(iii) Find a probability $\widetilde{\mathbb{P}}$, equivalent to $\mathbb{P}$, under which the inverse process X^{-1} is a local martingale.

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Solution to Exercise 2

From Itô's formula, we can write, for any $t \geq 0$,

$$X_t = \exp\left\{\int_0^t \left(\mu_s - \frac{1}{2}\sigma_s^2\right) \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}W_s\right\},$$

so that (i) follows immediately. One can apply Girsanov theorem to introduce the probability measure $\mathbb{Q}$ via $\mathrm{d}\mathbb{Q} = Z_t \mathrm{d}\mathbb{P}$ with $\mathrm{d}Z_t = Z_t \mu_t \sigma_t^{-1} \mathrm{d}W_t$, such that $B_t := W_t + \mu_t \sigma_t^{-1} \mathrm{d}t$ is a standard Brownian motion under $\mathbb{Q}$. Finally, applying Itô's formula yields

$$\mathrm{d}X_t^{-1} = -X_t^{-1}\sigma_t \left(\mathrm{d}W_t - \frac{\sigma_t^2 - \mu_t}{\sigma_t}\mathrm{d}t\right),$$

and (iii) follows again by a direct application of Girsanov's theorem.

Exercise 3 Ornstein-Uhlenbeck

Fix some $\lambda \in \mathbb{R}$, and let X be an Ornstein-Uhlenbeck process, e.g. the solution to

$$\mathrm{d}X_t = -\lambda X_t \mathrm{d}t + \mathrm{d}W_t, \qquad X_0 = 0 \in \mathbb{R}, \tag{0.1}$$

and introduce the process Z as

$$Z_t := \exp\left\{\lambda \int_0^t X_s \mathrm{d}W_s - \frac{\lambda^2}{2}\int_0^t X_s^2 \mathrm{d}s\right\}.$$

(i) Show that Z is a local martingale, and we shall from now on accept that it is a true martingale.

(ii) Define the new probability measure $\mathbb{Q}$ as $\mathrm{d}\mathbb{Q} := Z_t \mathrm{d}\mathbb{P}$. Write the stochastic differential equation solved by the process X under $\mathbb{Q}$.

(iii) Show that

$$Z_t := \exp\left\{\lambda \int_0^t X_s \mathrm{d}X_s + \frac{\lambda^2}{2}\int_0^t X_s^2 \mathrm{d}s\right\},$$

and compute, for any $u \in \mathbb{R}$, the expectation

$$\mathbb{E}^{\mathbb{P}}\left(\exp\left\{-\frac{u^2}{2}\int_0^t X_s^2 \mathrm{d}s\right\}\right).$$

You might need to show first that an application of Itô's formula yields

$$\int_0^t W_s \mathrm{d}W_s = \frac{W_t^2 - t}{2}.$$

Solution to Exercise 3

This is a straightforward application of Girsanov's theorem: Under $\mathbb{Q}$ the process $W_t - \lambda\int_0^t X_s \mathrm{d}s$ is a standard Brownian motion. Combining this with the SDE (0.1), we obtain that

$$X_t = x - \lambda\int_0^t X_s \mathrm{d}s + W_t$$

is a standard Brownian motion under $\mathbb{Q}$. Finally,

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left(\exp\left\{-\frac{u^2}{2}\int_0^t X_s^2\mathrm{d}s\right\}\right) &= \mathbb{E}^{\mathbb{Q}}\left(Z_t^{-1}\exp\left\{-\frac{u^2}{2}\int_0^t X_s^2\mathrm{d}s\right\}\right)\\
&= \mathbb{E}^{\mathbb{Q}}\left(\exp\left\{-\frac{u^2+\lambda^2}{2}\int_0^t X_s^2\mathrm{d}s - \lambda\int_0^t X_s\mathrm{d}X_s\right\}\right)\\
&= \mathbb{E}^{\mathbb{Q}}\left(\exp\left\{-\frac{u^2+\lambda^2}{2}\int_0^t X_s^2\mathrm{d}s - \frac{\lambda}{2}\left(X_t^2-t\right)\right\}\right)\\
&= \mathrm{e}^{\lambda t/2}\mathcal{N}\left(\frac{\lambda}{2},\sqrt{\lambda^2+u^2}\right),
\end{aligned}$$

where $\mathcal{N}$ denote the Gaussian cumulative distribution function.

Exercise 4 Application of Girsanov to Put-Call symmetry

Let S be a martingale satisfying the stochastic differential equation $\mathrm{d}S_t = \sigma S_t \mathrm{d}W_t$, starting from $S_0 = 1$, where σ is a strictly positive constant.

(i) Check that S_t is strictly positive almost surely for all $t \geq 0$.
(ii) Compute explicitly $X_t := S_t^{-1}$.
(iii) Let $\mathbb{Q}$ be a new probability measure defined via $\mathrm{d}\mathbb{Q} := S_t\mathrm{d}\mathbb{P}$. What is the law of X_t under $\mathbb{Q}$?
(iv) Show finally the Put-Call symmetry (different from the Put-Call parity!!!!):

$$\mathbb{E}^{\mathbb{P}}(S_T-K)_+ = K\mathbb{E}^{\mathbb{Q}}\left[\left(K^{-1}-X_T\right)_+\right],\qquad \text{for all } K>0.$$

Solution to Exercise 4

(i) Itô's lemma implies that $S_t = \exp\left(-\frac{1}{2}\sigma^2 t + \sigma W_t\right)$ for any $t \geq 0$. Since the Brownian motion does not explode to infinity over any finite time horizon, the result follows.
(ii) Using the previous representation, we immediately have

$$X_t = S_t^{-1} = \exp\left(\frac{1}{2}\sigma^2 t - \sigma W_t\right).$$

It further satisfies the stochastic differential equation (by Itô's lemma):

$$\mathrm{d}X_t = -\frac{\mathrm{d}S_t}{S_t^2} + \frac{\mathrm{d}\langle S_t\rangle}{S_t^3} = -\sigma X_t\mathrm{d}W_t + \sigma^2 X_t\mathrm{d}t.$$

(iii) Since $(S_t)_{t\geq 0}$ is a true strictly positive martingale, $\mathbb{Q}$ is a well-defined probability measure, equivalent to $\mathbb{P}$. Therefore the process $(B_t)_{t\geq 0}$ defined by $B_t := W_t - \sigma t$ is a standard Brownian motion under $\mathbb{Q}$, and so is $W^{\mathbb{Q}} := -B$, and hence

$$\mathrm{d}X_t = -\sigma X_t\left(\mathrm{d}W_t - \sigma\mathrm{d}t\right) = \sigma X_t \mathrm{d}W_t^{\mathbb{Q}}.$$

Under $\mathbb{Q}$, the process X is therefore a geometric Brownian motion.
(iv) Using the change of measure introduced previously, we can write, for all $K > 0$,

$$\mathbb{E}^{\mathbb{P}}(S_T-K)_+ = \mathbb{E}^{\mathbb{P}}\left[S_T\left(1-\frac{K}{S_T}\right)_+\right] = K\mathbb{E}^{\mathbb{Q}}\left[\left(\frac{1}{K}-\frac{1}{S_T}\right)_+\right] = K\mathbb{E}^{\mathbb{Q}}\left[\left(K^{-1}-X_T\right)_+\right].$$

Exercise 5 Asian option

Consider the process $(S_t)_{t\geq 0}$ given by

$$S_t = S_0 \exp\left(2\mu t + 2W_t\right).$$

Show that S is a submartingale whenever $\mu \geq -1$, and a supermartingale otherwise. Show then that the price of an Asian option, with payoff $(\frac{1}{T}\int_0^T S_u \mathrm{d}u - K)_+$ is greater than the corresponding Call price with payoff $(S_T - K)_+$, for small enough $K \geq 0$.

Hint: You may want to show first that the representation $S_t = S_0 +$ martingale $+ 2(1+\mu)\int_0^t S_u \mathrm{d}u$ holds almost surely for all $t \geq 0$, and then the trivial inequality (which follows from the convexity of the exponential function) $\mathrm{e}^x \leq 1 + x\mathrm{e}^x$ for any $x \in \mathbb{R}$.

Exercise 6 CEV Case

For any $\beta \in \mathbb{R}$, consider the process $(S_t)_{t\geq 0}$ defined as the solution to the stochastic differential equation

$$\mathrm{d}S_t = \sigma S_t^\beta \mathrm{d}W_t, \qquad S_0 = 1.$$

(1) What is this process when $\beta = 0$ and $\beta = 1$?

(2) Take $\sigma = 0.1$ and $\beta = 2$. Using the closed-form representation given in the lecture notes (Exercise 2.1.15), compute, on the same plot, the functions $K \mapsto \mathbb{E}(S_T - K)_+$, for $T \in \{0, 0.1, 1, 5\}$, and discuss the plots.

Exercise 7 Barrier option

Consider an up-and-out Barrier Call option, as in Section 1.4.2 in the notes. Using Proposition 1.4.14, and assuming the Black-Scholes model with volatility $\sigma = 20\%$ and constant interest rate $r = 4\%$,

(1) Implement the price of the barrier option;

(2) Discuss (with graphs) the influence of the barrier, in particular with respect to the corresponding standard Call option (no barrier);

(3) Compute the price of an 'up-and-in' Call option (you may use some smart symmetry with the 'up-and-out' case).

Department of Mathematics, Imperial College London

E-mail address: `a.jacquier@imperial.ac.uk`